Turbulent diffusion of gas bubbles in large-scale homogeneous isotropic turbulence

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Abstract

In this paper the turbulent diffusion of gas bubbles with large Reynolds number is investigated analytically. The turbulence is supposed to be statistically homogeneous, isotropic and stationary. A fairly well-established equation of motion of the bubbles is approximated for large-scale turbulence. This approximate equation of motion is used to derive a closed set of equations for the turbulent dispersion tensor components of bubbles in a way analogeous to that derived by Saffman (1963) for fluid particles and that by Pismen & Nir (1978) for small solid particles. The results are compared with results obtained from numerical simulations reported in a previous paper.

1 Introduction

The motion of a gas bubble moving with velocity $\mathbf{V}(t)$ through a turbulent flow $\mathbf{U}(\mathbf{x}, t)$ is governed by the equation of motion

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathcal{F}(\mathbf{V}(t), \mathbf{U}(\mathbf{X}(t), t)), \tag{1}$$

where $\mathbf{X}(t)$ denotes the position of the bubble, and $\mathcal{F}$ is a nonlinear functional of the fluid and bubble velocities. To obtain statistical properties of the bubble motion such as the mean rise velocity of bubbles one has to solve this equation, supposed that it is known, and express these statistical properties in terms of those of the fluid along the unknown path of the bubble. Statistical data of turbulent flows are usually only available from measurements at fixed positions. So one faces the problem of relating Lagrangian statistics with Eulerian statistics. This problem occurs also when instead of bubbles marked fluid elements or solid particles are considered. An approximate way to relate Lagrangian to Eulerian statistics in the case of fluid particles has been proposed by Corrsin (1959). 'Corrsin's conjecture' was used subsequently by Saffman (1963) who obtained a closed equation for the turbulent dispersion of fluid particles. Pismen & Nir (1978) extended Saffman's analysis for the dispersion tensor of small solid particles.

More recently it has been shown that solid particles may adopt a averaged fall velocity that differs significantly from the settling velocity in still fluid (see e.g. Wang & Maxey 1993). The drift velocity of particles, made dimensionless with the r.m.s. fluid velocity, has been shown by Nir & Pismen (1979), and earlier by Csanady (1963) to be an important parameter in the turbulent diffusion of solid particles. For bubbles, it is therefore important to relate their averaged rise velocity to known quantities such as the rise velocity in still fluid. This is done in the next section of this paper for large-scale turbulence. The bubble motion is taken to be governed by the equation of motion proposed by Thomas, Auton, Sene & Hunt (1983).

In section 3 a closed set of equations for components of the dispersion tensor is derived, closely following Pismen & Nir's (1978) analysis for solid particles. In section 4 results are presented and compared with results from numerical simulations that are presented in another paper (Spelt & Biesheuvel 1997).

2 Mean rise velocity

The proposal of Thomas *et al.* (1984) is to use the following equation of motion

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = 3\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} - (\mathbf{V}-\mathbf{U})\times\mathbf{\Omega} - 2\mathbf{g} - \frac{1}{\tau_b}(\mathbf{V}-\mathbf{U}), \tag{2}$$

where the time constant τ_b is defined by $\tau_b = a^2/(18\nu) = V_T/(2g)$, $\mathbf{V}(t)$ is the bubble velocity, $\mathbf{U}(\mathbf{X}(t),t)$ the ambient fluid velocity at the position of the bubble and V_T the rise velocity in still fluid. This equation has been used by Sene, Hunt & Thomas to study the interaction of gas bubbles with coherent vortices in a plunging water jet. For want of anything better this equation is adopted here as well.

To calculate the mean rise velocity of bubbles in large-scale turbulence, the calculation by Maxey (1987) of the averaged fall velocity of solid particles with a small relaxation time can be followed. When $\mathcal{L}$ and $\mathcal{U}$ are a characteristic length- and velocity scale of the fluid velocity field, and $\tau_b V_T/\mathcal{L}$ is supposed to be small, the fluctuating bubble velocity is

$$\mathbf{V} \;=\; \mathbf{V}_T + \mathbf{u} + O\left(\mathcal{U}\frac{\tau_b V_T}{\mathcal{L}}\right). \tag{3}$$

The next higher-order approximation is the 'bubble velocity field':

$$\mathbf{V}^*(\mathbf{x},t) = \mathbf{V}_T + \mathbf{u} - \tau_b\left(\frac{\partial}{\partial t} + (\mathbf{V}_T+\mathbf{u})\cdot\nabla\right)\mathbf{u} + 3\tau_b\frac{D\mathbf{u}}{Dt} - \tau_b\mathbf{V}_T\times(\nabla\times\mathbf{u}), \tag{4}$$

correct to $O(\mathcal{U}(\tau_b V_T/\mathcal{L})^2)$. This expression can be rewritten as

$$\mathbf{V}^*(\mathbf{x},t) = \mathbf{V}_T + \mathbf{u} + \tau_b\left(2\frac{D\mathbf{u}}{Dt} - \mathbf{V}_T\cdot\nabla\mathbf{u} - \mathbf{V}_T\times(\nabla\times\mathbf{u})\right). \tag{5}$$

Within this approximation, the velocity of a bubble depends only on the local circumstances, i.e. not on the history of the bubble motion. By taking the

divergence of (5),

$$\nabla \cdot \mathbf{V}^*(\mathbf{x},t) = 2\tau_b \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \tau_b V_T \nabla^2 u_x(\mathbf{x},t), \tag{6}$$

it can be seen by which regions in turbulent flows bubbles are attracted; this happens where the divergence (6) has a negative value. Hunt, Wray & Moin (1988) introduced a classification of characteristic zones in turbulent flows. They identify 'eddy zones' as regions in the flow where the pressure and the second invariant of the rate-of-deformation tensor, defined as

$$\Pi = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = e_{ij}e_{ij} - \tfrac{1}{2}\omega_k^2,$$

with e_{ij} the rate-of-strain tensor and ω_k the vorticity, are less than some negative treshold value. The first term on the right-hand side of (6) being strongly negative coincides partly with this definition of eddy zones. That bubbles can be trapped by vortices has been investigated by Sene, Hunt & Thomas (1995). Maxey (1987) found for solid particles the same term but with a positive sign (and no factor 2), indicating that particles collect in zones of the flow with a high strain rate and low vorticity and are repelled by eddy zones.

The second term on the right-hand side of (6) arises from the action of lift forces; it also arises when considering low values of β. Spelt & Biesheuvel (1997) showed that lift forces make bubbles move in vertical shear towards regions with minimum or downward fluid velocity.

Next, Maxey's (1987) averaging technique can be used to find the lowest-order deviation of the mean bubble rise velocity from V_T:

$$\begin{aligned} \overline{V} - V_T &= -2\tau_b \int_0^\infty \overline{u_x(\mathbf{x},t)\Pi(\mathbf{X}(\mathbf{x},t;\tau),\tau)}\mathrm{d}\tau \\ &\quad + \tau_b V_T \int_0^\infty \overline{u_x(\mathbf{x},t)(\nabla^2 u_x)(\mathbf{X}(\mathbf{x},t;\tau),\tau)}\mathrm{d}\tau. \end{aligned} \tag{7}$$

This is correct to $O(\mathcal{U}(\tau_b V_T/\mathcal{L})^2)$. $\mathbf{X}(\mathbf{x},\tau)$ is the bubble path that satisfies (3) and $\mathbf{X}(\mathbf{x},\, \tau = t) = \mathbf{x}$. Although this is not a practically applicable expression for $\overline{V} - V_T$, several conclusions can be drawn from it. It is seen that for smaller values of $\mathcal{U}/V_T$, the largest contribution to $\overline{V} - V_T$ arises from the action of lift forces, i.e., from the last term in (7), which is of $O(\mathcal{U}\tau_b V_T/\mathcal{L})$. For larger values of β the trapping of bubbles in vortices gives rise to the one-but-last term in (7), which is of $O(\mathcal{U}\tau_b\mathcal{U}/\mathcal{L})$. In either case though the deviation of $\overline{V}$ from V_T is small in large-scale turbulence: the bubbles rise on average in large-scale turbulence approximately with their settling velocity in still fluid.

3 A closed equation for the dispersion of the bubbles

Pismen & Nir (1978) presented a calculation of the dispersion tensor of solid particles

$$\overline{[X_i(t) - \overline{X}_i(t)][X_j(t) - \overline{X}_j(t)]}$$

for an arbitrary value of $\beta \equiv u_0/V_T$ (with u_0 the r.m.s. value of the fluctuation of one component of the fluid velocity). Components of this tensor can be related to the autocorrelation function of the fluctuation in the bubble velocity

$$\lim_{t_0\to\infty} \overline{[V_i(t_0) - \overline{V}_i][V_j(t_0+t) - \overline{V}_j]} = \lim_{t_0\to\infty} \overline{v_i(t_0)v_j(t_0+t)};$$

by (Taylor 1921, Batchelor 1949):

$$\begin{aligned} &\overline{[X_i(t) - \overline{X_i}(t)][X_j(t) - \overline{X_j}(t)]} \; = \\ &\qquad \lim_{t_0\to\infty} \int_0^t (t-\tau) \left(\overline{v_i(t_0)v_j(t_0+\tau)} + \overline{v_j(t_0)v_i(t_0+\tau)} \right) \mathrm{d}\tau. \end{aligned} \tag{8}$$

For long times τ the diffusion process is characterized by the diffusion coefficients

$$\mathcal{D}_i \;=\; \lim_{t\to\infty} \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \overline{[X_i(t) - \overline{X}_i(t)][X_j(t) - \overline{X}_j(t)]}, \;=\; \int_0^\infty \overline{v_i(t_0)v_j(t_0+t)} \mathrm{d}t. \tag{9}$$

The more complex form of the equation of motion of a bubble as compared to that of a small solid partcile (especially the presence of the lift force) makes a calculation of the dispersion of bubbles similar to that of Pismen & Nir (1978) very difficult. But it is possible to approximate the equation of motion of a bubble for large-scale or low-energy turbulence by an equation of motion for which the dispersion can be calculated straightforwardly in an analogous way to that of solid particles.

A simplified equation of motion of a bubble

The assumptions that led to the equation of motion (2) of a bubble can be used to approximate this equation as follows. The equation of motion can be rewritten in the following form, with $\mathbf{W} = \mathbf{U} - \mathbf{V}$:

$$\begin{aligned} -\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} &= -\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} - 2\mathbf{g} + \frac{1}{\tau_b}\mathbf{W} + 3\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} + \mathbf{W}\times\Omega \\ &= \mathbf{W}\cdot\nabla\mathbf{U} + 2\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}t} - 2\mathbf{g} + \frac{1}{\tau_b}\mathbf{W} + \mathbf{W}\times\Omega. \end{aligned}$$

Multiplication with a/W^2, using the notation $W = |\mathbf{W}|$, gives

$$-\frac{a}{W^2}\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} = 3\frac{a}{W}\frac{\mathbf{W}}{W}\cdot\nabla\mathbf{U} + 2\frac{a}{W^2}\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} - \frac{2a}{W^2}\mathbf{g} + \frac{a}{\tau_b W^2}\mathbf{W} + \frac{a}{W^2}\mathbf{W}\times\Omega, \tag{10}$$

with $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \mathbf{V}\cdot\nabla$. Except for the buoyancy and drag terms, all the terms in this equation are explicitly assumed to be small in the derivation of (2). So the relative velocity is

$$\mathbf{W} = -\mathbf{V}_T + O\left(\frac{a}{W}||\nabla\mathbf{U}|| \left(\frac{W\tau_b}{a}\right) W, \frac{a}{W^2}\left|\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t}\right| \left(\frac{\tau_b W}{a}\right) W \right). \tag{11}$$

This expression can be substituted in (10) to give an approximate equation of motion that can be written as

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = -3\mathbf{V}_T\cdot\nabla\mathbf{U} + 3\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} - \mathbf{V}_T\times\mathbf{\Omega} - 2\mathbf{g} - \frac{1}{\tau_b}(\mathbf{V}-\mathbf{U}). \tag{12}$$

The error introduced by using this equation instead of (2) is a factor $W\tau_b/a$ larger than the error that is present in (2). The error in (2) becomes small if velocity gradients in the turbulence become small, so (12) is a good approximation of (2) for large-scale or low-energy turbulence. It can be compared with equation (4) above for large-scale turbulence. The left-hand side of (12) is approximated with the third term on the right-hand side in (4). On the other hand, the fourth term on the right-hand side of (4) is approximated in (12) with the first two terms on the right-hand side; this causes no error larger than already present in (4).

An expression for bubble velocity correlations

An expression for bubble velocity correlations is obtained from the simplified equation of motion (12) by writing it as

$$\begin{aligned}\mathbf{v}(t) = \mathbf{V}(t) - \overline{\mathbf{V}} &= 3\mathbf{u}(\mathbf{X}(t),t) - \int_0^t \mathrm{d}\tau e^{-|t-\tau|/\tau_b} \{3\mathbf{V}_T \cdot \nabla \mathbf{u}(\mathbf{X}(\tau),\tau) \\ &\quad + \mathbf{V}_T \times (\nabla \times \mathbf{u}(\mathbf{X}(\tau),\tau)) + \frac{2}{\tau_b}\mathbf{u}(\mathbf{X}(\tau),\tau)\Big\}. \qquad (13)\end{aligned}$$

In this equation, $\mathbf{u}(\mathbf{X}(t),t)$ is the fluid velocity at the position of the bubble at time t minus the mean fluid velocity along the bubble path; $\nabla\mathbf{u}(\mathbf{X}(t),t)$ is the fluid velocity gradient at the position of the bubble at time t minus the mean fluid velocity gradient along the bubble path. This gives for the bubble velocity correlation

$$\begin{aligned}\overline{v_i(t_0)v_j(t_0+t)} &= 9\overline{u_i(\mathbf{X}(t_0),t_0)u_j(\mathbf{X}(t_0+t),t_0+t)} \\ &-3\int_0^{t_0} e^{-|t_0-t'|/\tau_b}\overline{I_i(\mathbf{X}(t'),t')u_j(\mathbf{X}(t_0+t),t_0+t)}\mathrm{d}\tau \\ &-3\int_0^{t_0+t} e^{-|t_0+t-t'|/\tau_b}\overline{u_i(\mathbf{X}(t_0),t_0)I_j(\mathbf{X}(t'),t')}\mathrm{d}\tau \\ &+\int_0^{t_0}\int_0^{t_0+t} e^{-|t_0-t'|/\tau_b-|t_0+t-t''|/\tau_b}\overline{I_i(\mathbf{X}(t'),t')I_j(\mathbf{X}(t''),t'')}\mathrm{d}t''\mathrm{d}t' \qquad (14)\end{aligned}$$

with

$$I_i = \frac{2}{\tau_b}u_i + 3\mathbf{V}_T \cdot \nabla u_i + \epsilon_{ijk}\omega_k V_{T,j}. \qquad (15)$$

The velocity correlation tensor is axisymmetric, the axis of symmetry being aligned with the gravity direction. Hence it will suffice to calculate only two components of the dispersion tensor; say, the autocorrelation of the vertical velocity component and the sum of all three autocorrelations of the velocity components in each direction. Taking the limit $t_0 \to \infty$ gives the following expressions for these correlations:

$$\begin{aligned}\lim_{t_0\to\infty}\overline{v_x(t_0)v_x(t_0+t)} &= 9\lim_{t_0\to\infty}\overline{u_x(\mathbf{X}(t_0),t_0)u_x(\mathbf{X}(t_0+t),t_0+t)} \\ &+9V_T\lim_{t_0\to\infty}\int_t^\infty e^{-|t'-t|/\tau_b}\frac{\partial}{\partial r_x}\overline{u_x(\mathbf{X}(t_0),t_0)u_x(\mathbf{X}(t_0+t'),t_0+t')}\mathrm{d}t'\end{aligned}$$

$$
\begin{aligned}
&-9V_T \lim_{t_0\to\infty} \int_{-\infty}^{t} e^{-|t'-t|/\tau_b} \frac{\partial}{\partial r_x} \overline{u_x(\mathbf{X}(t_0),t_0)u_x(\mathbf{X}(t_0+t'),t_0+t')}\mathrm{d}t' \\
&+ \lim_{t_0\to\infty} \int_{-\infty}^{\infty} e^{-|t'-t|/\tau_b} \left\{ -\frac{4}{\tau_b} \overline{u_x(\mathbf{X}(t_0),t_0)u_x(\mathbf{X}(t_0+t'),t_0+t')} \right. \\
&\left. -\tfrac{9}{2}\tau_b V_T^2 \frac{\partial^2}{\partial r_x^2} \overline{u_x(\mathbf{X}(t_0),t_0)u_x(\mathbf{X}(t_0+t'),t_0+t')} \right\} \mathrm{d}t', \qquad (16)
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{t_0\to\infty} \overline{v_i(t_0)v_i(t_0+t)} &= 9 \lim_{t_0\to\infty} \overline{u_i(\mathbf{X}(t_0),t_0)u_i(\mathbf{X}(t_0+t),t_0+t)} \\
&+6V_T \lim_{t_0\to\infty} \int_{t}^{\infty} e^{-|t'-t|/\tau_b} \frac{\partial}{\partial r_x} \overline{u_i(\mathbf{X}(t_0),t_0)u_i(\mathbf{X}(t_0+t'),t_0+t')}\mathrm{d}t' \\
&-6V_T \lim_{t_0\to\infty} \int_{-\infty}^{t} e^{-|t'-t|/\tau_b} \frac{\partial}{\partial r_x} \overline{u_i(\mathbf{X}(t_0),t_0)u_i(\mathbf{X}(t_0+t'),t_0+t')}\mathrm{d}t' \\
&- \lim_{t_0\to\infty} \int_{-\infty}^{\infty} e^{-|t'-t|/\tau_b} \left\{ \frac{4}{\tau_b} \overline{u_i(\mathbf{X}(t_0),t_0)u_i(\mathbf{X}(t_0+t'),t_0+t')} \right. \\
&+2\tau_b V_T^2 \frac{\partial^2}{\partial r_x^2} \overline{u_i(\mathbf{X}(t_0),t_0)u_i(\mathbf{X}(t_0+t'),t_0+t')} \\
&\left. +\tfrac{1}{2}\tau_b V_T^2 \frac{\partial^2}{\partial r_i^2} \overline{u_x(\mathbf{X}(t_0),t_0)u_x(\mathbf{X}(t_0+t'),t_0+t')} \right\} \mathrm{d}t' \qquad (17)
\end{aligned}
$$

where there is summed over i.

A closed set of equations for the dispersion

The dispersion tensor follows from the velocity correlation tensor by (8):

$$
\begin{aligned}
&\overline{[X_i(t)-\overline{X_i}(t)][X_j(t)-\overline{X_j}(t)]} = \\
&\qquad \lim_{t_0\to\infty} \int_0^t (t-\tau) \left(\overline{v_i(t_0)v_j(t_0+\tau)} + \overline{v_j(t_0)v_i(t_0+\tau)} \right) \mathrm{d}\tau. \qquad (18)
\end{aligned}
$$

A closed set of equations for the dispersion tensor can now be obtained by substituting the expressions (16)-(17) for the bubble velocity correlations in the right-hand side of (8) and subsequently using Corrsin's conjecture.

Corrsin's conjecture is that for large times the displacement vector $\mathbf{X}(t)$ is statistically independent of the local fluid velocity $\mathbf{u}(\mathbf{X}(t))$. This conjecture means that (Saffman 1963)

$$
\begin{aligned}
\overline{\mathbf{u}(0,0)\mathbf{u}(\mathbf{X}(t),t)} &\cong \int_{-\infty}^{\infty} \overline{\mathbf{u}(\mathbf{0},0)\mathbf{u}(\mathbf{r},t)}\ \overline{\delta(\mathbf{r}-\mathbf{X}(t))}\mathrm{d}\mathbf{r} \\
&= \int_{-\infty}^{\infty} \mathbf{\Phi}(\mathbf{k},t)\overline{\exp\left[-i\mathbf{k}\cdot\mathbf{X}(t)\right]}\mathrm{d}\mathbf{k}, \qquad (19)
\end{aligned}
$$

where $\mathbf{\Phi}(\mathbf{k},t)$ is the usual Fourier transform of the *Eulerian* velocity correlation. This approximation was discussed by Saffman (1963) for fluid particles. Saffman argued that this approximation does not account for a bias, if present, in the fluid velocity sampled by a particle out of an ensemble of realizations of the flow.

It is shown in Spelt & Biesheuvel (1997) that in turbulence with intermediate length scales, bubbles sample on average a fluid downflow, so that it is not expected that (19) is a good approximation in that case. We have seen above, though, that the preferential concentration of bubbles in certain zones of the turbulence vanishes for all values of β if large-scale motion is considered (cf. equation (6)) and for all length scales if small values of β are considered. So it is expected that Corrsin's conjecture can be used for large-scale turbulence and for low-energy turbulence. This coincides with the conditions under which the approximation (12) of the equation of motion of bubbles was made above.

Similar expressions for gradients of the velocity correlation that are sampled by the bubble (and occur in the expressions for the bubble velocity correlations written down above) are straightforwardly derived. In line with Corrsin's conjecture one assumes that $\mathbf{X}(t)$ has a Gaussian probability density function, so that

$$\overline{\exp\left[-\mathrm{i}\mathbf{k}\cdot\mathbf{X}(t)\right]} = \cos(\mathbf{k}\cdot\overline{\mathbf{X}(t)})\,\exp\left[-\tfrac{1}{2}\mathbf{kk}\colon\overline{\left[\mathbf{X}(t)-\overline{\mathbf{X}(t)}\right]\left[\mathbf{X}(t)-\overline{\mathbf{X}(t)}\right]}\right]. \tag{20}$$

So by combining (8) with (19), (20), (16) and (17), a set of integro-differential equations for the mean-square displacement is obtained; only the energy-spectrum tensor needs to be specified. Following the analysis of Saffman (1963) and Nir & Pismen (1979) the energy-spectrum tensor is taken as

$$\Phi_{ij}(\mathbf{k},t) \;=\; \frac{E(k)}{4\pi k^2}\left(\delta_{ij} \;-\; \frac{k_i k_j}{k^2}\right)\exp(-\tfrac{1}{2}a^{*2}u_0^2k^2t^2) \tag{21}$$

with the following energy spectrum function

$$E(k) \;=\; 16\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{u_0^2k^4}{k_0^5}\exp(-2k^2/k_0^2).$$

The parameter a^* in (21) is taken here equal to 0.5.

The dispersion tensor is axisymmetric, the axis of symmetry being aligned with the gravity direction. Hence it will suffice to calculate only two components of the dispersion tensor; say, its projection on the unit vector in the upwards direction, defined as $\mathbf{p}$, and its projection on a unit vector in the plane normal to $\mathbf{p}$. Writing

$$\mathbf{kk} \;=\; A(k,x)\mathbf{I} \;+\; B(k,x)\mathbf{pp},$$

with $x = \mathbf{k}\cdot\mathbf{p}/k$, one finds that $A(k,x) = \frac{1}{2}(1-x^2)k^2$ and $B(k,x) = -\frac{1}{2}(1-3x^2)k^2$. It is convenient to introduce the following dimensionless quantities:

$$y = k_0^2\overline{[X_i(t)-\overline{X_i}(t)][X_i(t)-\overline{X_i}(t)]},$$

$$y_3 = k_0^2\overline{[X_x(t)-\overline{X_x}(t)][X_x(t)-\overline{X_x}(t)]}, \quad \tau = u_0k_0t$$

Carrying out the integration in (19) and in similar expressions for correlations of gradients of fluid velocity finally yields

$$\frac{\mathrm{d}^2y}{\mathrm{d}\tau^2} = \frac{144}{a^{1/2}c}\left(\frac{1}{2c}-\frac{a^2}{32d^2}+\frac{1}{8d}\right)\exp\left[-\frac{a^2}{8d}\right]$$

$$
\begin{aligned}
&+\int_{-\infty}^{\infty} \mathrm{e}^{-\lambda_1^*|\tau-\tau'|}\left\{-\frac{64\lambda_1^*}{d'^{1/2}c'}\left(\frac{1}{2c'}-\frac{a'^2}{32d'^2}+\frac{1}{8d'}\right)-\frac{3a'}{\beta_1 d'^{3/2}c'}\left(\frac{4}{c'}-\frac{a'^2}{4d'^2}+\frac{3}{d'}\right)\right.\\
&+\frac{1}{\lambda_1^*{\beta_1}^2 d'^{3/2}c'}\left(\frac{a'^4}{16d'^3}-\frac{3a'^2}{2d'^2}+\frac{3}{d'}-\frac{a'^2}{d'c'}+\frac{4}{c'}\right)\\
&\left.-\frac{1}{2\lambda_1^*{\beta_1}^2 d'^{1/2}c'^2}\left(\frac{a'^2}{4d'^2}-\frac{1}{d'}-\frac{8}{c'}\right)\right\}\times\exp\left[-\frac{a'^2}{8d'}\right]\,\mathrm{d}\tau'\\
&+\int_{-\infty}^{\tau} \mathrm{e}^{-\lambda_1^*|\tau-\tau'|}\frac{6a'}{\beta_1 d'^{3/2}c'}\exp\left[-\frac{a'^2}{8d'}\right]\left(-\frac{a'^2}{4d'^2}+\frac{3}{d'}+\frac{4}{c'}\right)\mathrm{d}\tau', \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2 y_3}{\mathrm{d}\tau^2} &= \frac{36}{d^{1/2}c^2}\exp\left[-\frac{a^2}{8d}\right]-\int_{-\infty}^{\infty} \mathrm{e}^{-\lambda_1^*|\tau-\tau'|}\left\{\frac{16\lambda_1^*}{d'^{1/2}c'^2}\right.\\
&\left.+\frac{9}{4\lambda_1^*{\beta_1}^2 d'^{3/2}c'^2}\left(\frac{a'^2}{2d'}-2\right)+\frac{9a'}{\beta_1 d'^{3/2}c'^2}\right\}\times\exp\left[-\frac{a'^2}{8d'}\right]\,\mathrm{d}\tau'\\
&+\int_{-\infty}^{\tau} \mathrm{e}^{-\lambda_1^*|\tau-\tau'|}\frac{18a'}{\beta_1 d'^{3/2}c'^2}\exp\left[-\frac{a'^2}{8d'}\right]\mathrm{d}\tau' \qquad (23)
\end{aligned}
$$

where the following notation has been used:

$$a=\tau/\beta_1, \quad d=1+\tfrac{1}{4}(a^*\tau)^2+\tfrac{1}{4}y_3(\tau), \quad c=2+\tfrac{1}{4}(a^*\tau)^2+\tfrac{1}{4}(y(\tau)-y_3(\tau));$$

a' refers to a at time τ' instead of at τ, etc. The problem is characterized by the dimensionless parameters

$$\lambda_1^*=\frac{1}{\tau_b u_0 k_0}, \quad \mu_1^*=\frac{1}{\tau_b \overline{V} k_0}, \quad \beta_1=\frac{\mu_1^*}{\lambda_1^*}=\frac{u_0}{\overline{V}}. \qquad (24)$$

Equations (22) and (23) are the equivalents of equation (9) in the paper by Nir & Pismen (1979). These equations can be written as a system of integral equations for y and y_3 is partly of the Fredholm type as well as partly of the Volterra type. They were solved by discretizing y and y_3 in the time domain and approximating the integrals using the Euler integration scheme. The result of this discretization is a set of algebraic non-linear equations that can be solved using a root-finding routine (Baker 1977, chapter 6). The Newton-Raphson root-finding method can be used for this purpose. This procedure was tested on the related problems solved by Saffman (1963) and Nir & Pismen (1979), but because of the complexity of the current set of equations a simpler root-finding method is used here. The old estimates of y and y_3 are simply substituted in the right-hand sides of the integral equations corresponding to equations (22)-(23); this gives new values for y_3 and y at all times. The resulting iteration process always converged in a few steps, and was found not to be sensitive to chosen initial values.

4 Discussion of results

The longitudinal bubble diffusivity $\mathcal{D}_x$, normalized by $u_0 L_{11}$, is plotted in figure 1*(a)*, and the mean-squared value of the bubble velocity fluctuation component in the longitudinal direction $\overline{v_x^2}$ in figure 1*(b)*, for $1 \le \mu_1^* \le 6$ and $0 < \beta_1 \le 5$.

For values of β_1 smaller than unity $\mathcal{D}_x$ decreases much faster with β_1 than $u_0 L_{11}$ to zero. This sharp decrease is explained by the crossing trajectories effect: bubbles move approximately along a straight path through the turbulence, crossing the trajectories of fluid particles, which results in short correlation times of the fluctuating fluid velocity field sampled by the bubbles. In figure 2, the obtained bubble diffusivities and mean-square bubble velocity fluctuations are plotted together with the following analytical results that were derived in Spelt & Biesheuvel (1997) for the asymptotic case $\beta_1 \ll 1$:

$$\mathcal{D}_x = \beta u_0 L_{11} + O(\beta^2 u_0 L_{11}), \tag{25}$$

$$\mathcal{D}_y = \tfrac{1}{2}\beta u_0 L_{11} + \tfrac{\pi}{4}\tau_b^2 V_T \int_0^\infty kE(k)\mathrm{d}k + O(\beta^2 u_0 L_{11}), \tag{26}$$

$$\overline{v_x^2} = \int_0^\infty E(k)\phi_x(k\tau_b V_T)\mathrm{d}k + O(\beta u_0^2), \tag{27}$$

$$\overline{v_y^2} = \int_0^\infty E(k)\phi_y(k\tau_b V_T)\mathrm{d}k + O(\beta u_0^2), \tag{28}$$

with

$$\phi_x(x) = -\frac{1}{x^2} + \left(\frac{1}{x^3} + \frac{1}{x}\right)\tan^{-1}(x),$$

$$\phi_y(x) = \frac{1}{2} + \frac{1}{2x^2} - \left(\frac{1}{x} + \frac{1}{x^3} - \frac{x}{2}\left(\frac{1}{x^2} + 1\right)^2\right)\tan^{-1}(x),$$

The present results agree well with these asymptotic results, even for values of β_1 up to $\beta_1 = 0.25$.

For larger values of β_1 figure 2*(a)* for $\mu_1^* = 2$ shows that the increase in dimensionless diffusivities vanishes; both (dimensionless) diffusivities are even seen to decrease if β_1 becomes 1. Figure 1*(a)* shows that this effect vanishes if μ_1^* is increased.

As each diffusivity can be written as the product of $\overline{v_i^2}$ and the integral timescale $\mathcal{T}_i$, and the mean-square bubble velocity fluctuations still increase, this decrease in $\mathcal{D}_i$ is due to a strong decrease of the integral timescale of the bubble velocity fluctuation when β_1 increases. The reason for this sharp decrease can be found by studying bubble velocity correlations presented in figure 3 for several values of β_1. The decrease of integral timescales of the bubble velocity components (and therefore of the bubble diffusivities) at larger values of β_1 is seen from this figure to be caused by negative loops in both bubble velocity correlations. These negative loops indicate that the sign of a bubble velocity fluctuation is typically reversed, which naturally inhibits turbulent diffusion.

The cause of these negative loops of the bubble velocity correlations can be investigated by studying results for a 'bubble' with the equation of motion

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = 3\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} - 2\mathbf{g} - \frac{1}{\tau_b}(\mathbf{V} - \mathbf{U}). \tag{29}$$

Velocity correlations resulting from equation (29) are plotted in figure 3*(b)*. The results when using this equation of motion are quite similar to those presented here, especially for not so small values of β_1, i.e., when the negative loops do

occur in the bubble velocity correlations. These negative loops for the results of equation of motion (29) can be explained as follows. The expression analogous to equation (14) for the 'bubble' velocity correlation is for this equation of motion:

$$\lim_{t_0\to\infty} \overline{v_i(t_0)v_j(t_0+t)} = 9 \lim_{t_0\to\infty} \overline{u_i(\mathbf{X}(t_0),t_0)u_j(\mathbf{X}(t_0+t),t_0+t)} - \lim_{t_0\to\infty} \frac{4}{\tau_b}\int_{-\infty}^{\infty} e^{-|t'-t|/\tau_b}\overline{u_i(\mathbf{X}(t_0),t_0)u_j(\mathbf{X}(t_0+t'),t_0+t')}\mathrm{d}t'. \quad (30)$$

The negative loops occur especially at larger values of β_1 and lower values of μ_1^*. This suggests to consider the case that τ_b is much larger than any other timescale (such as L_{11}/u_0). The exponent in the integral in (30) can then be approximated by $\exp(-t/\tau_b)$, so that

$$\lim_{t_0\to\infty} \overline{v_i(t_0)v_j(t_0+t)} \approx 9 \lim_{t_0\to\infty} \overline{u_i(\mathbf{X}(t_0),t_0)u_j(\mathbf{X}(t_0+t),t_0+t)} - \lim_{t_0\to\infty} \frac{4}{\tau_b}e^{-t/\tau_b}\int_{-\infty}^{\infty} \overline{u_i(\mathbf{X}(t_0),t_0)u_j(\mathbf{X}(t_0+t'),t_0+t')}\mathrm{d}t' \quad (31)$$

The fluid velocity along the 'bubble' path is correlated no longer significantly after a time $\sim L_{11}/u_0$, so that the first term on the right-hand side of (31) vanishes after $t \sim L_{11}/u_0$, while the second term vanishes only after time $t \sim \tau_b \gg L_{11}/u_0$. So after $t \sim L_{11}/u_0$ the (negative) second term dominates; this causes a negative loop in 'bubble' velocity correlations with long time-lags. The similarity between the results from both (29) and (12) suggests an analogous explanation of the negative loops found when using the latter. Expression (31) also indicates that if τ_b becomes large with respect to other timescales (such as L_{11}/u_0) the mean-squared velocity fluctuation in any direction, governed by equation of motion (29), approaches $9u_0^2$. This is confirmed by figure 1*(a)*, which shows that $\overline{v_x^2}$ increases to $9u_0^2$ if μ_1^* decreases and if β increases.

Comparison with numerical simulation results

Numerical simulations of the motion of bubbles in a turbulent flow are presented in Spelt & Biesheuvel (1997). The technique that has been used to simulate turbulent velocity fields is called Kinematic Simulation and is an improved version of that presented by Kraichnan (1970). The energy spectrum function of the simulated turbulent flow is prescribed in KS; this makes a comparison possible with the analytical calculations above.

Figure 4 compares results from numerical simulations when the equation of motion (2) is used with the results when (12) is used. Although $\lambda^* = 1$, which is not large-scale turbulence, the differences are small and only quantitative: the simplified equation just gives slightly larger diffusivities and bubble velocity fluctuations. So for the case $\lambda^* \geq 1$ the simplified equation of motion used in the analysis ((equation (12)) can be used instead of (2).

In figure 2, the obtained bubble diffusivities and the mean-square bubble velocity fluctuations are compared with the numerical simulation results as a function of $\beta_1 \equiv u_0/\overline{V}$. The numerical simulation results were obtained for

$\lambda^* = \lambda/(\tau_b V_T) = 10$; the resulting values of $\overline{V}$ were used to calculate the values of $\beta_1 = u_0/\overline{V}$ and $\mu_1^* = \lambda/(2\tau_b\overline{V})$ that are input parameters of the analytical calculations. The dependence on the value of μ_1^* was found to be small for large values of μ_1^*; for convenience in figure 5 the analytical results are plotted for $\mu_1^* = 5$. It was shown by Lundgren & Pointin (1976) that KS and the analysis of Saffman (1963) that was used as a basis in the analytical calculations give the same diffusivity for fluid particles. For bubbles there is good agreement between the results if $\beta_1 \ll 1$, but several differences are seen in figure 5 for not asymptotically small β_1, although the order of magnitude of the analytically obtained diffusivities is correct.

The strong increase in dimensionless longitudinal diffusivity observed in the numerical simulation results at $\beta \leq 0.4$ is seen not to be predicted by the present analysis. Also, the strong decrease in dimensionless longitudinal diffusivity at higher values of β is not well reproduced by the analysis. These trends coincide with the lack of increase in the mean-square longitudinal bubble velocity fluctuation component and an overprediction of $\overline{v^2}$ at larger values of β.

In the previous section it is mentioned that Corrsin's conjecture does not take account for possible biased fluid velocity sampled by the bubbles. In section 2 it was shown that such a bias does occur when length scales in the turbulence are not large compared to $\tau_b V_T$. So if the observed differences between numerical and analytical results are due to the bias in fluid velocity statistics sampled by the bubbles, this means that the case of $\lambda^* = 10$ gives length scales in the turbulence that are not large enough for the present analysis to apply. This is confirmed by the fact that the peak in the dimensionless longitudinal diffusivity at $\lambda^* = 4$ was found to be about 0.65 high, while at $\lambda^* = 10$ it is seen in figure 5*(a)* to be a little over 0.5, clearly showing a tendency toward the analytical result.

Appendix: Low β large lengthscales

There is the possibility that there is so much energy contained in the large length scales of the turbulence that (**??**) is no longer satisfied and the analysis above is not applicable. In this situation the relaxation time of the bubbles is much less than the characteristic time scales of the turbulence, i.e. $\lambda^*, \mu^* \gg 1$. This case was also considered in section **??** for solid particles. In the (second) analysis presented there (which is due to Maxey 1987a) one assumes that condition (**??**) is met.

The largest contribution to (**??**) is of $O(\mathcal{L}/\tau_b\mathcal{U})$; neglecting the other terms gives case (a) mentioned in the beginning of this section:

$$v_i' = u_i',$$

or

$$\mathbf{V} \;=\; \mathbf{V}_T + \mathbf{u} + O\left(\mathcal{U}\frac{\tau_b V_T}{\mathcal{L}}\right), \tag{32}$$

which is the same as for solid particles. Unlike the lowest-order result for intermediate-length scale turbulence, this does not represent a straight line; it

are deviations of the bubble motion from merely following the fluid (superposed to the straight path) that are considered here rather than deviations from a straight path.

The next higher-order approximation can be found by following the (second) derivation presented in section **??**, but now with the equation of motion (**??**), where the time t has been made dimensionless with $\mathcal{L}/V_T$ rather than with $\mathcal{L}/u_0$ as in section **??**. This gives the following 'bubble velocity field':

$$\mathbf{V}^*(\mathbf{x},t) = \mathbf{V}_T + \mathbf{u} - \tau_b \mathbf{V}_T \cdot \nabla \mathbf{u} - \tau_b \mathbf{V}_T \times (\nabla \times \mathbf{u}), \tag{33}$$

correct to $O(\beta V_T(\tau_b V_T/\mathcal{L})^2, \beta^2 V_T(\tau_b V_T/\mathcal{L}))$: within this approximation, the velocity of a bubble still depends only on the local circumstances, i.e. not on the history of the bubble motion. Expression (33) is the equivalent for bubbles of equation (**??**) for solid particles. By taking the divergence of (33),

$$\nabla \cdot \mathbf{V}^*(\mathbf{x},t) = -\tau_b \nabla \cdot (\mathbf{V}_T \times (\nabla \times \mathbf{u}))\,, = -\tau_b V_T \nabla^2 u_x(\mathbf{x},t), \tag{34}$$

it can be seen by which regions in turbulent flows bubbles are attracted; this happens where the divergence (34) has a negative value. It is this attraction to certain zones in the turbulence that the fluid velocity sampled by bubbles becomes non-zero on average, which causes a deviation in the mean rise velocity of the bubbles. The divergence (34) is negative in regions where $\nabla^2 u_x > 0$. This quantity can be integrated over a region where this is the case,

$$\int \nabla^2 u_x \mathrm{d}V = \int \nabla u_x \cdot \mathbf{n} \mathrm{d}A, \tag{35}$$

with $\mathbf{n}$ the outward normal of the surface A of the volume V. This integral is positive for the regions considered, so that it follows that these regions have a local minimum of the vertical velocity. So bubbles are attracted, due to the lift force, by regions in the flow where the vertical component of the fluid velocity has a local minimum. This can be associated with the action of lift forces depicted in figure **??**.

Next, Maxey's (1987a) averaging technique (see section **??**) can be used to find the lowest-order deviation of the mean bubble rise velocity from V_T:

$$\overline{V} - V_T = 4\beta^2 V_T \int_0^\infty \frac{1}{r^*} \frac{\partial f}{\partial r^*}(r^*, r^*) \mathrm{d}r^*. \tag{36}$$

This is correct to $O(\beta V_T(\tau_b V_T/\mathcal{L})^2, \beta^3 V_T(\tau_b V_T/\mathcal{L}))$; the second error estimate is smaller than it was in expression (33) because the $O(\beta^2 V_T(\tau_b V_T/\mathcal{L}))$-term has been found to give no contribution to the mean rise velocity.

The result coincides with (**??**), obtained for intermediate length scales, and thus extending the applicability of the result of that analysis. The deviation of $\overline{V}$ from V_T is of much smaller magnitude here than in the case of intermediate length scales above as it decreases with $\tau_b V_T/\mathcal{L}$. The order of magnitude of $\overline{V} - V_T$ given by (36) is the same as in the corresponding case of solid particles discussed in section **??**. The average fluid velocity along the bubble path differs from that found before; it can readily be shown that $\overline{u} = \overline{V} - V_T$ when only

large length scales are present in the turbulence. The results differ further when evaluating the higher-order contributions.

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Figure 1.
Longitudinal bubble diffusivity *(a)*, normalized by $u_0 L_{11}$, and the longitudinal mean-square bubble velocity fluctuation *(b)*, divided by u_0^2, as functions of β_1 and μ_1^*.

Figure 2.
(a) Longitudinal and lateral bubble velocity diffusivities with $\mu_1^* = 2$ using the simplified equation of motion. — —, longitudinal diffusivity (——, asymptotical result (25) for small β_1); – - –, lateral diffusivity (– – –, asymptotical result (25) for small β_1). *(b)* Mean square bubble velocity fluctuations. — —, longitudinal (——, asymptotical result (27) for small β_1); – - –, lateral (– – –, asymptotical result (28) for small β_1).

Figure 3.
Longitudinal (——) and lateral (– – –) bubble velocity correlations with $\mu_1^* = 2$, using *(a)* equation (12) *(b)* equation (29). Numbers refer to values of β_1.

Figure 4.
(a) Longitudinal and lateral bubble velocity diffusivities using either the full or the simplified equation of motion (open and filled symbols respectively). Kraichnan spectrum with $\lambda^* = 4$. □, longitudinal diffusivity; ◯, lateral diffusivity. *(b)* Mean square bubble velocity fluctuations. Symbols as in *(a)*.

Figure 5.
(a) Longitudinal and lateral bubble velocity diffusivities and the simplified equation of motion. Kraichnan spectrum $\lambda^* = 10$. Lines refer to analytical results with $\mu_1^* = 5$, markers to numerical simulation results. □, ——, longitudinal diffusivity; ◯, – – –, lateral diffusivity. *(b)* Mean square bubble velocity fluctuations. Longitudinal: numerical simulations (□) and analytical result of chapter 3 (——); lateral: numerical simulations (◯) and analytical result of chapter 3 with $\mu_1^* = 5$ (– – –).