Creeping flows of power-law fluids through periodic arrays of elliptical cylinders

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Abstract

Results from numerical simulations and lubrication theory are presented for creeping flows of power-law fluids through periodic arrays of elliptical cylinders. Flows are considered in the plane perpendicular to the axes of the cylinders, both along an axis of the array (on-axis flow) and at an angle to the axes of the array (off-axis flows). Results are presented for the apparent permeability tensor and for the dimensionless velocity variances (which can also be used to approximate the added mass coefficient for a cylinder in the array). The apparent permeability values obtained for on-axis flows of power-law fluids are shown to obey a simple scaling, which relates the apparent permeability tensor for power-law fluids to the corresponding permeability for Newtonian fluids; this scaling arises because of the choice of length scale used in the definition of the apparent permeability tensor for power-law fluids. The off-axis flow results are shown to be related to the on-axis results in a straightforward manner. The results are summarised in the form of closure relations for the apparent permeability tensor and velocity variances for off-axis flows of power-law fluids through arrays of elliptical cylinders for a range of aspect ratios using look-up graphs for only a few scalars.

Keywords: power-law fluids; permeability; elliptical cylinders

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1. Introduction

The flow through periodic arrays of aligned cylinders has been studied extensively, especially for Newtonian fluids [1-3], because of its importance in heat exchanger design and the manufacturing of fibre-reinforced composites, and as a model problem for the flow through porous media encountered in oil recovery processes. The main issue is the determination of the permeability coefficient tensor which, for creeping flows of Newtonian fluids through an infinite periodic array of cylinders (of characteristic dimension *a*), can be introduced from a scaling argument in combination with a force balance. First, the drag force per unit length $\boldsymbol{F}$ acting on a cylinder in the array can be scaled with the superficial fluid velocity $\langle \boldsymbol{V} \rangle$ (the velocity averaged over the whole area, with zero velocity inside the cylinders) and the fluid viscosity μ:

$$F_i \equiv \frac{\mu}{n_d a^2} K_{ij}^{-1}(\phi)\left\langle V_j \right\rangle, \tag{1}$$

where K_{ij} is a permeability tensor made dimensionless by the length *a* (for example the radius when investigating circular cylinders) that depends on the (cross-sectional) solid area fraction ϕ ; n_d is the number of cylinders per unit cross-sectional area ($\phi \propto n_d a^2$). Due to the linearity of the Stokes equations for Newtonian fluids K_{ij}^{-1} does not depend on the direction of the superficial fluid velocity. For steady homogeneous flow through a uniform bed of cylinders the force $-n_d\boldsymbol{F}$ acting on the fluid per unit cross-sectional area is balanced by the pressure gradient in the fluid [4, see also (5) below]. Substituting the scaling relation (1) in this force balance leads to Darcy's law for large-scale flow through porous media

$$\left\langle V_i \right\rangle = -\frac{K_{ij}^{*}}{\mu}\frac{\partial P}{\partial x_j}, \tag{2}$$

where ∇P is the averaged pressure gradient in the porous medium and $K_{ij}^{*} = a^2 K_{ij}$. We note that the relations (1)-(2) are also valid for flows with finite inertia (see *e.g.* Koch and Ladd [3]), but in this case K_{ij} depends on the direction of the superficial fluid velocity, as well as the Reynolds number $Re \equiv \rho a \left|\langle \boldsymbol{V} \rangle\right| / \mu$. The permeability is commonly defined as a strictly geometrical factor that does not allow a dependence on Reynolds number or flow direction, so we shall refer to K_{ij} (defined by (1)) as the apparent permeability to avoid confusion. The apparent permeability coefficients K_{ij} can be obtained from (1) using numerical simulations; here we apply a specified pressure drop over a unit cell in the array (from which $\boldsymbol{F}$ is readily obtained), then integrate the equations of motion to a steady state and subsequently calculate $\langle \boldsymbol{V} \rangle$.

In this paper we study slow flows of non-Newtonian fluids through arrays of cylinders. The case of circular cylinders has been studied previously (numerically and analytically) by Bruschke and Advani [5] and Spelt *et al.* [6, an extended version has been submitted for publication]. As an alternative, Vijaysri, Chhabra and Eswaran [7] used a cell model rather than a periodic array, as a representation of a random array of aligned cylinders. In these studies a (truncated) power-law model was used for the fluid rheology. This slow flow limit corresponds to flows for which the Reynolds and Deborah numbers are sufficiently small that elastic and inertial effects can be neglected. (If the applied pressure drop over an array exceeds a critical value the resulting flow rate decreases dramatically due to elastic instabilities [8,9].)

For non-Newtonian fluids the scaling (1) has to be replaced (and (2) has to be changed accordingly) because the viscosity is no longer independent of shear rate. Furthermore, a proportionality factor similar to K_{ij} will depend on the flow direction and (dimensionless) rheological parameters. We choose a typical shear rate magnitude $\left|\langle V_i \rangle\right|/a$ so, for power-law fluids of index n and coefficient k, the viscosity in (1) can be replaced by a characteristic viscosity scale. The obvious scale to use would be $ka^{1-n}\left|\langle \boldsymbol{V} \rangle\right|^{n-1}$, but we have found that this leads to a strong dependence of the apparent permeability coefficients on the direction of the flow. Here, we use anisotropic scales $ka^{1-n}\left|\langle V_i \rangle\right|^{n-1}$, for each direction i, which give

$$F_i = (n_d a^2)^{-1} ka^{1-n} K_i^{-1}(n,\phi,\theta)\left|\langle V_i \rangle\right|^{n-1}\langle V_i \rangle, \quad (3)$$

where no summation over i is presumed. The notation makes it explicit that for non-Newtonian fluid flow, the permeability coefficients can depend on the angle θ that the superficial fluid velocity makes with an axis of the array. Here we assume that the cylinders and the array have sufficient symmetry that flow in one coordinate direction does not cause a force on the cylinder in the other direction. For off-axis flows, the dependence of F_i on the other component of velocity (e.g. of F_1 on V_2) is accounted for by the dependence of K_i on θ; this dependence is found to be weak. We shall refer to the K_i as the apparent permeability coefficients, because we can substitute the scaling (3) into the large-scale force balance to give a version of Darcy's law for power-law fluids in which the K_i play the role of the permeability coefficients,

$$\left|\langle V_i \rangle\right|^{n-1} \langle V_i \rangle = -k^{-1} a^{1+n} K_i(n,\phi,\theta) \frac{\partial P}{\partial x_i}. \tag{4}$$

The usual (and equally valid) choice of the magnitude of the superficial fluid velocity to the power n-1 on the left-hand side [10] could have been obtained by using the velocity magnitude rather than $\left|\langle V_i \rangle\right|$ in the scaling of the viscosity that led to (3), but we shall see that the form of (3) leads to only a small θ-dependence of the coefficients K_i.

In the studies of flow through arrays of circular cylinders [5,6] the apparent permeability coefficients K_i have been shown to depend very strongly on the power-law index but this dependency could be scaled out by using a particular length (and velocity) scale to estimate the typical value of the apparent viscosity in (3). This scaling has been demonstrated for a wide range of cylinder arrangements and flow angles, including square as well as hexagonal arrays of cylinders, flow along the cylinders and even, to some extent, for the first effects of fluid inertia [6]. As a result of this scaling argument (which we shall present in more detail in Section 3 below) the apparent permeability for flow of a power-law fluid at an angle θ to an axis of the array could be related to the permeability for on-axis flow of a Newtonian fluid [6]. In [6] the results by Tanner [11] for shear-thinning fluid flow past a single cylinder were shown to follow similar behaviour.

A motivation of this study is that in many applications, cylinders of circular cross-section are an idealisation. For example, in the resin transfer moulding process for fibre-reinforced composites the fibre bundles are usually compressed, leading to a more elliptical shape [12]. Therefore, in this paper we study the more complicated geometry of periodic arrays of elliptical cylinders. The question to be addressed following the previous work on circular cylinders is whether results for cylinders of non-circular cross-section also obey a simple scaling relation, or are circular cylinders a degenerate case? The effect of cylinder aspect ratio (as well as cylinder arrangement) for the case of Newtonian fluids was shown by Larson and Higdon [13] to have a strong effect on the (anisotropic) apparent permeability tensor. A strong quantitative dependence on aspect ratio was also found by Wang [14] for periodic arrays of rectangular cylinders. For non-Newtonian fluids a further complication is that the apparent permeability coefficients depend on the flow angle because the equations of motion are not linear. For flows that are not aligned with one of the axes of the array one might expect the non-linearity of the equations of motion to cause significant deviations from any simple scaling relation. However, if a simple scaling relationship could be found, it would be very useful for the modelling of non-Newtonian fluid flows through porous media.

In this article, we study the problem of inelastic flows of a non-Newtonian fluid through an array of elliptical cylinders, for flow in the plane perpendicular to the cylinders (transverse flow). The flow may be along one of the axes of the array (on-axis flow) or not (off-axis flow).

The results presented in this paper can thus be used for simulations that are based on (a modified) Darcy's law, but they are also more widely applicable. One improvement on Darcy's law is to use a full description of the flow through a porous medium. For saturated flows, Darcy's law (Eq. (2)) can be replaced by [4]

$$\frac{\partial\langle V_j\rangle}{\partial x_j}=0,\qquad \rho\frac{\partial\langle V_i\rangle}{\partial t}+\rho\frac{\partial}{\partial x_j}\left(\langle V_i\rangle\langle V_j\rangle\right)=n_d\langle F_i\rangle-\rho\frac{\partial}{\partial x_j}\langle v_i v_j\rangle-\frac{\partial P}{\partial x_j}+\frac{\partial\langle\tau_{ij}\rangle}{\partial x_j}. \tag{5}$$

The brackets indicate locally (volume- or ensemble-) averaged values; $v_i=V_i-\langle V_i\rangle$ is the deviation of the local velocity from the mean, τ_{ij} is the stress tensor and n_d is the number of cylinders per unit cross-sectional area. Darcy's law can be obtained from the above momentum equation by omitting all temporal and spatial derivatives, apart from the pressure gradient; by setting $n_d=0$ the usual Reynolds-averaged Navier-Stokes equations for fluid flows are obtained. For this improved description the permeability is required (for the closure of the first term on the right-hand side of (6)), and the velocity variances $\langle v_i v_j\rangle$ also have to be closed, i.e., replaced by some function of known variables, such as $\langle V_i\rangle$. These Reynolds stress terms become significant compared to the viscous term (which is maintained in the commonly used Brinkman approximation) if the large-scale Reynolds number $\rho UL/\mu$ (with U and L a velocity and length scale of the large-scale motion) is of $O(1)$ or larger. This does not require that the flow on the scale of individual cylinders must be beyond creeping flow, as pointed out by Koch and Hill [15]: if there is a sufficient separation of length scales in a flow, effects of inertia may be important in the large-scale motion even if they are not in the small-scale motion. The steady-state velocity variance components can be obtained along with the steady-state drag force. However, closure relations for the averaged viscous stress tensor $\langle\tau_{ij}\rangle$ cannot be obtained from the simulations presented here for uniform flow through a periodic array of cylinders. For that a shear flow through the array would have to be simulated, which would lead, in an infinite medium, to large fluid velocities relative to the cylinders. An obvious approximate closure for power-law fluids would be to use the same dependence of the stress on the (averaged) rate-of-strain tensor.

Recently Hill, Koch and Ladd [16] have shown that the unsteady drag force on a particle in a periodic array in an accelerating flow can be approximated by adding the added mass force to the steady-state drag force, and that in the large-time limit the added mass coefficient can be related, by using an argument based on (5), to the steady-state dimensionless velocity variances in the array. For these reasons we present in this paper closure relations for the velocity variances as well as results for the apparent permeability coefficients.

2. Problem definition and numerical method

The general equations of creeping motion of incompressible fluids are

$$\frac{\partial V_j}{\partial x_j} = 0, \qquad \rho \frac{\partial V_i}{\partial t} = -\frac{\partial P}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j}, \tag{7}$$

where V_i is the i-component of the fluid velocity vector, ρ is the density of the fluid, P is the pressure and τ_{ij} is the ij-component of the viscous part of the stress tensor. The rheology of many inelastic, non-Newtonian fluids can be approximated by the power-law fluid model. For power-law fluids the viscous part of the stress tensor is related to velocity gradients through

$$\tau_{ij} = 2k\Pi^{(n-1)/2} E_{ij}, \qquad E_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right), \tag{8}$$

where $\Pi=2E_{lm}E_{lm}$ is the second invariant of the rate-of-strain tensor E_{ij} (summation over the indices l and m is presumed); k and n are the power-law coefficient and power-law index, respectively. We shall consider values of the power-law index between 0.5 and 1.5. We have followed here the common practice of using the truncated power-law model [17] instead such that the effective viscosity does not exceed a specified maximum value, to avoid numerical problems in points or lines of symmetry. We have verified that the maximum viscosity value used was so large that its value did not affect the results presented here.

A schematic diagram of the geometry for which these equations have been solved is shown in Figure 1. Larson and Higdon [13] presented results for Newtonian fluids flowing through square arrays of elliptical cylinders, for which maximum packing is reached at a relatively low area fraction. In order to avoid this, the aspect ratio of the unit cell is set equal to that of the cylinders in the current work giving a single aspect ratio $A=b/a$. Starting from rest, a (known) pressure drop is applied over this unit cell of the periodic array and the equations of motion are integrated to a steady state wherein the pressure drop is balanced by the force exerted by the cylinder on the fluid. Periodic boundary conditions are used, with the pressure anywhere on the right face being equal to that at the corresponding point on the left face of the unit cell minus the specified pressure drop over the cell. Different flow angles can be achieved by applying a pressure drop in both horizontal and vertical directions. The steady-state superficial fluid velocity $\langle \boldsymbol{V} \rangle$ is then obtained by integrating the velocity vector components over the entire unit cell (the velocity in the cylinder interior is zero); each component of the force $\boldsymbol{F}$ is obtained from the pressure drop in that direction times the relevant dimension of the unit cell. In this paper we present results mainly for the case A=0.2. The case of circular cylinders (A=1) has been studied previously [6].

The method that was used to solve the equations of motion for this problem is essentially that proposed by Zang, Street and Koseff [18] for Newtonian fluids. This is a finite-difference, fractional step method in which the equations of motion have been written in terms of generalised coordinates, and has been used for a wide range of problems (see e.g., Zang and Street [19]). A non-staggered mesh is used; a typical mesh is shown in Figure 2. The only difference between our method and that of Zang, Street and Koseff [18] arises from the different rheology of power-law fluids. We have included the calculation of the second invariant Π at cell faces in a consistent manner by using the neighbouring cell-centered velocity values (as described in [6]). We shall compare our results against analytical results for limiting cases.

3. Results and Discussion

3.1. Apparent permeability

For the arrays of elliptical cylinders we define the apparent permeability coefficients K_i by

$$F_i = (\pi / \phi) k a^{1-n} K_i^{-1}(n, \phi, A, \theta) \left| \langle V_i \rangle \right|^{n-1} \langle V_i \rangle \, , \tag{9}$$

in which there is no summation over the index *i*. Because of the symmetry of the cylinder cross-sections and the array, no lift is generated for on-axis flow. The results for cylinders of aspect ratio A can also be used for those with aspect ratio A^{-1} through the relationship $K_1(A^{-1}) = A^{1-n} K_2(A)$. For n=1, K_1 and K_2 do not depend on the direction of the fluid flow and are equal to their respective on-axis values. (This was used to check the simulation results for Newtonian fluid flows, where the variation of K_1 and K_2 with flow angle was found to be less than 0.6%.)

3.1.1. On-axis flows

We first consider flows along the major axis (θ=0) and the minor axis (θ=π/2) for A=0.2. Fig. 3 shows the permeability coefficients obtained for Newtonian fluids for a range of solid area fractions ϕ. The results show higher permeability values for flow along the minor axis than for flow along the major axis. This is due to the wider flow path in the direction of the minor axis: for flow along the major axis the fluid has to squeeze through a much narrower gap to traverse the unit cell, hence there is a larger drag force. For the same reason the permeability decreases with increasing solid area fraction.

The Newtonian permeability coefficients are compared in Fig. 3 with results from a lubrication theory, details of which are given in the Appendix. At large area fractions the main contribution to the pressure drop over a unit cell arises from the narrow gap between the cylinders. The lubrication result for the permeability along the major axis through an array of elliptical cylinders is

$$K_1 = (8\pi)^{-1/2} A^n \left(\frac{\pi}{\phi}\right) \left(\frac{n}{2n+1}\right)^n \frac{\Gamma(2n+1)}{\Gamma\left(2n+\frac{1}{2}\right)} \left(1-\left(\frac{\phi}{\phi_{\max}}\right)^{1/2}\right)^{2n+\frac{1}{2}} \qquad (\phi_{\max}-\phi \ll 1), \qquad (10)$$

where $\phi_{\max} = \pi/4$ is the maximum packing solid fraction (which is independent of A because the unit cell and the cylinders are taken to have the same aspect ratio). The numerical simulation results for K_1 in Figure 3 are seen to agree very well with Eq. (10) (with n=1), even at rather low area fractions. The range of validity of the lubrication theory is not restricted (as it is for arrays of circular cylinders) to high values of ϕ because of the slender cross section of the elliptical cylinders, which makes the flow almost unidirectional in the entire unit cell. The same result (10) can be used for the flow along the minor axis (using $K_2(A) = A^{n-1} K_1(A^{-1})$), but it is clear from Figure 3 that there is a significant deviation with the numerical simulation results in the solid fraction range shown. The assumption in the lubrication theory that the pressure drop occurs mainly in the narrow gap between the cylinders requires a solid fraction close to maximum packing for flow along the minor axis.

Fig. 4 shows the apparent permeability coefficients obtained for non-Newtonian fluids flowing through a periodic array for a range of solid area fractions. Again, the numerical results for K_1 are seen to agree well with Eq. (10) (shown by the solid lines in Fig. 4a). The apparent permeabilities depend significantly on the power-law index as well as on the solid area fraction. In [6] the apparent permeability for arrays of circular cylinders was found to exhibit similar behaviour; this was shown to arise from the way in which the apparent permeability is defined (results in [6] were actually presented in terms of the drag coefficient, $C_d = \pi\phi^{-1}K_1^{-1}$). We have used here the definition (9) for the apparent permeability, which is an obvious extension of the definition for Newtonian fluids. For power-law fluids the viscosity depends upon the local shear rate in the fluid through an algebraic relation (8). Hence the viscosity has been replaced by a combination of terms including the power-law coefficient, a length scale and a velocity scale: in the definition (9) we have used the superficial fluid velocity and a dimension of the cylinders. In [6] we showed that for the case of circular cylinders it is possible to choose the length and velocity scale in such a way that the dependence of the apparent permeability on the power-law index is virtually eliminated. In order to examine whether the same can be done for the elliptical cylinders considered here, we introduce a length scale L and velocity scale $U_L \equiv \langle V_i \rangle c/L$ (c is the dimension of the unit cell in Figure 1). We can introduce new apparent permeability coefficients based on these scales (following the argument that led to (3)) and choose L such that the new apparent permeabilities do not depend on the power-law index. The new apparent permeability coefficients can be related to K_i. The ratio of the new coefficient for power-law coefficient n to that for n=1 then results in the following expression for $L(\phi, n)$

$$\frac{L}{a} = A^{\frac{1}{2}}\left(\frac{\phi_{\max}}{\phi}\right)^{\frac{1}{4}}\left(\frac{K_i(\phi, n=1)}{K_i(\phi, n)}\right)^{\frac{1}{2-2n}}. \tag{11}$$

The numerical simulation results for L/a are shown in Fig. 5. L/a is indeed almost independent of the power-law index. The averaged value of L over n, denoted by $\mathcal{L}$, was found to depend very strongly on solid fraction. Therefore we present results in Fig. 6 for the ratio $\mathcal{L}/L_g$ as a function of solid area fraction, where $L_g = c - b$ is half the minimum size of the gap between the cylinders. $\mathcal{L}/L_g$ for flow along the major axis is seen to be very close to unity for all area fractions, whereas the minor-axis results for $\mathcal{L}$ are somewhat larger than L_g. The apparent permeability for on-axis flows is approximated well by the inverse of Eq. (11),

$$K_i(\phi, n) = K_i(\phi, n=1)A^{1-n}\left(\frac{\phi_{\max}}{\phi}\right)^{(1-n)/2}\left(\frac{\mathcal{L}}{a}\right)^{2n-2}, \tag{12}$$

which is shown by the dashed lines in Fig. 4. Alternatively, the lubrication theory (Eq. (10)) can be used for the case of on-axis flow along the major axis.

3.1.2. Off-axis flows

The simple scaling result Eq. (12) can be used for flows along one of the axes of the array. We now examine how the apparent permeability changes if a pressure drop is applied in both horizontal and vertical directions over a unit cell. The angles that the superficial fluid velocity and the drag force make with the major axis are denoted by θ and θ_F, respectively. The variation of K_1 and K_2 with θ_F is shown in Fig. 7. The transverse permeability shows a significant angular dependence only for nearly on-axis flow of non-Newtonian fluids. However, this is not significant for the development of closure relations, since that component will be multiplied by the very small velocity component (see Eq. (9)). That (9) is the most appropriate scaling is born out by the fact that if, rather than $\left|\left\langle V_i \right\rangle\right|$, the magnitude of the averaged velocity had been used to find a viscosity scale (cf. the discussion preceding (3)), then the (newly defined) apparent permeability coefficients would have been significantly dependent on the flow angle.

Due to the large difference between the apparent permeability for flow along the major and minor axes the angle that the flow makes with the minor axis will be smaller than that made by the pressure gradient. If the apparent permeability coefficients are fully independent of flow direction it follows from (9) that

$$\theta = \tan^{-1}\left(\left(\frac{K_2}{K_1} \tan \theta_F \right)^{1/n} \right). \tag{13}$$

We shall denote the right-hand side of (13) by $\Theta_{F.}$ From Fig. 8 we see that θ is not identical to Θ_F but that this hypothesis (Eq. (13)) is a reasonable approximation. Furthermore, θ is seen to be a function of the stretched angle Θ_F only, not of area fraction. In conclusion, the closure (Eq. (9)) with the on-axis values of K_1 and K_2 gives sufficient accuracy for practical purposes.

3.1.3. Arrays of cylinders of intermediate aspect ratio.

We have shown in the above that the apparent permeability coefficients for off-axis flows of power-law fluids can be obtained with reasonable accuracy from the corresponding on-axis flow values. Furthermore, we have shown that these on-axis values for power-law fluids can be obtained from the corresponding values for Newtonian fluids by using Eq. (12) with $\mathcal{L}$ approximated by half the minimum gap size through which the fluid is forced. We have shown this to be the case for A=0.2 and for A=1 (elsewhere [6]). We assume without proof that this remains true for intermediate values of A. The relationship between $K_1(A)$ and $K_2(A^{-1})$ can then be used to extend the results to the range A>1. All one needs then, to obtain the apparent permeability components for off-axis flows of power-law fluids are the permeability coefficients for on-axis flows of Newtonian fluids. In Figure 9, we present these results for a range of aspect ratios. We see that the results for K_1 and K_2 are continuous through A=1, as required, and that they increase rapidly with aspect ratio, especially at low values of A (see also the result from lubrication theory (10) above). We note that the reduced minimum gap size between the cylinders (divided by the cell height) does not decrease with aspect ratio. Rather, at small A, the gap size is small over a larger region, which leads to a larger drag force and a smaller permeability value.

3.2. *Velocity variances*

As pointed out in the Introduction it is of interest to include results for the velocity variance components, as these are required when using the full equations of motion rather than Darcy's law for the description of the large-scale flow through fibrous media, as well as for the determination of the unsteady force on a cylinder in an array.

3.2.1. *On-axis flows*

For on-axis flows, we define the dimensionless velocity variance components R_{ij}^{σ} by

$$\left\langle \left(V_i - \langle V_i \rangle\right)\left(V_j - \langle V_j \rangle\right) \right\rangle \equiv R_{ij}^{\sigma}(n,\phi)\langle V_\sigma \rangle^2 , \tag{14}$$

where no summation is presumed over i or j and brackets indicate cell-averaged values. For flow along the major axis σ=1 and for flow along the minor axis σ=2; for off-axis flows, a different scaling will be used. Because of symmetry, $R_{12}^{\sigma} = 0$ Fig. 10 shows the components R_{11}^{1} and R_{22}^{2} for flow along the major and minor axes as functions of ϕ and n. These velocity variances show a strong dependence on solid area fraction, but not on n. A high area fraction corresponds to a small gap size between the cylinders which results in a larger variation in fluid velocity through the unit cell, hence the dependence on solid fraction. Also shown in Fig. 10 is the result of the analytical (lubrication) theory, details of which are given in the Appendix:

$$R_{11}^{\ 1} = 2^{1/2}\pi \frac{(1+2n)}{(2+3n)}\left(1-\left(\frac{\phi}{\phi_{\max}}\right)^{1/2}\right)^{-1/2} \qquad (\phi_{\max}-\phi \ll 1), \tag{15}$$

which is independent of aspect ratio, and here also serves as an approximation for R_{22}^{2}.

In the case of flow along the major axis, the fluid is flowing along a much more streamlined shape than in the case of flow along the minor axis, which results in a smaller velocity variance. Fig. 11 shows the velocity variance components transverse to the main flow direction. At high values of ϕ these exhibit some dependence on n but this is not significant in magnitude compared to the magnitude of the velocity variance component in the main-flow direction. We can thus, to a good approximation, use the Newtonian value of the velocity variances and neglect the transverse component. The velocity variance component in the main flow direction for Newtonian fluid flow is given in Figure 12 for a range of cylinder aspect ratios and area fractions. We see that at large solid fractions the velocity variances vary relatively little with aspect ratio, in agreement with the lubrication theory (15).

3.2.2. Off-axis flows

For off-axis flows we write the velocity variances as

$$\left\langle \left(V_i - \langle V_i \rangle\right)\left(V_j - \langle V_j \rangle\right)\right\rangle \equiv R_{ij}(n,\phi,\theta_F)\langle V_i \rangle\langle V_j \rangle + R_{11}{}^2\langle V_2 \rangle^2 \delta_{i1}\delta_{j1} + R_{22}{}^1\langle V_1 \rangle^2 \delta_{i2}\delta_{j2}\,, \qquad (16)$$

where no summation is presumed over i or j; δ_{ij} is the Kronecker delta. The last two terms represent the tendency of, for instance, $\langle V_2{}^2\rangle$ to approach its finite on-axis flow value in the case of small values of θ_F when $\langle V_2\rangle$ is small. Defining R_{ij} without these terms leads to a very large dependence on flow angle. Fig. 13 shows the dimensionless velocity variance components R_{ij} for off-axis flows as a function of the stretched angle that the drag force makes with the major axis, Θ_F (defined by Eq. (13)). Both R_{11} and R_{22} are seen to be virtually independent of Θ_F, so that we simply have $R_{11} \approx R_{11}{}^1$ and $R_{22} \approx R_{22}{}^2$. Also, the dependence on power-law index is seen to be weak: using the Newtonian result suffices. The cross-correlation R_{12} was found be very small (less than 0.04), and approximately independent of flow angle.

4. Conclusion

Numerical simulation results for the flow of power-law fluids through periodic arrays of elliptical cylinders have been presented, together with results from a lubrication theory for concentrated arrays. The flow is in the plane perpendicular to the cylinders. Previous work [5,6] on the corresponding case of circular cylinders has shown that the strong dependence of the apparent permeability coefficient of the array (defined by (3)) on the power-law index could approximately be scaled out, and the issue addressed in this paper is whether the same holds for the more complicated geometry of elliptical cylinders. We have shown that this is indeed the case and that, as a result, the apparent permeability coefficients for power-law fluid flow under any angle in the plane perpendicular to the cylinders can be obtained from the corresponding permeability for Newtonian fluids flowing along an axis of the array.

Recently velocity variances have been shown to be important for the determination of the unsteady force acting on fixed arrays of particles [16]. These quantities are also important for the closure of averaged equations of motion for flow through porous media, as argued in the introduction (following equation (5)). We have shown that, despite the anisotropy of the arrays of elliptical cylinders, the dimensionless velocity variances show a negligible dependence on flow angle, if defined carefully. They are also shown to be insensitive to the value of the power-law index. Hence the results for velocity variances could be summarised in a simple form for a range of aspect ratios and area fractions of the cylinders.

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Appendix: Lubrication Theory

For on-axis flow (along the major axis) through concentrated arrays the main contribution to the pressure drop over a unit cell occurs in the narrow gap between the elliptical cylinders. By calculating the flow in this narrow gap the drag force on a cylinder in the array can be determined. In this Appendix we derive explicit expressions for the apparent permeability, Eq. (10) and for the velocity variance (15). The analysis is a generalisation of the results of Keller [20] and Spelt *et al.* [6] for flow through arrays of circular cylinders of Newtonian and power-law fluids, respectively.

We consider the almost unidirectional flow of a power-law fluid through a channel of varying width, $2h(x)$. The momentum equation in the transverse (y-)direction shows that the pressure gradient is almost constant over the width, and may be denoted by $G(x)$. Ignoring the small transverse velocity component and spatial derivatives in the x-direction results in the solution of the x-momentum equation for the fluid velocity

$$V_1(x,y) = \frac{n}{1+n}\left|\frac{G(x)h^{n+1}(x)}{k}\right|^{1/n}\left(1-\left|\frac{y}{h(x)}\right|^{1+1/n}\right), \tag{17}$$

where $y=\pm h(x)$ corresponds to the channel walls. The condition that the total flow through the gap is independent of x results in the following expression for the pressure gradient,

$$G(x) = \gamma h^{-2n-1}(x). \tag{18}$$

The proportionality constant γ is obtained by integrating Eq. (18) over the entire gap length and equating the result to the pressure drop over the cell, ΔP. Before carrying out the integration the channel half-width is approximated by a polynomial, $h(x) \approx c - b + x^2 b/(2a^2)$; the error introduced by this is negligible for concentrated arrays, since the narrowest part of the gap is correctly approximated. The result is

$$\gamma = \frac{c^{2n+1}}{(2\pi)^{1/2} a}\frac{\Gamma(2n+1)}{\Gamma\left(2n+\frac{1}{2}\right)}\left(1-\frac{b}{c}\right)^{2n+\frac{1}{2}}\Delta P, \tag{19}$$

where we have left out terms that are of higher order in the small parameter ($1-b/c$), for consistency. The superficial fluid velocity, $\langle V_1\rangle$, follows from integrating the velocity over the height of the unit cell (using Eq. (17) in the fluid and zero inside the cylinders):

$$\langle V_1 \rangle = \frac{n}{1+2n} \frac{\gamma^{1/n}}{ck^{1/n}}. \qquad (20)$$

Eq. (10) for the permeability is obtained from the definition of permeability (Eq. (9)), with $\langle F_1 \rangle = 2c\Delta P$, together with Eqs. (19) and (20). Eq. (15) for the velocity variance is obtained by integrating the square of Eq. (17) over x and y.

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Figure captions

Fig. 1. Problem definition sketch for a unit cell of a periodic array of elliptical cylinders. The aspect ratio of the unit cell is the same as that of the cylinder.

Fig. 2. Typical mesh, with solid area fraction $\phi = 0.4$ and aspect ratio *A*=0.2. Only half of the mesh is shown.

Fig. 3. Variation of permeability coefficients with solid area fraction ϕ for flow of a Newtonian fluid through a rectangular array of elliptical cylinders along the major (☝) and minor (Δ) axes. The lubrication theory result (10) for K_1 and K_2 are indicated by the solid and dashed lines, respectively. The aspect ratio *A*=0.2.

Fig. 4. Variation of apparent permeability with power law index for flow along the major axis (a) and minor axis (b) for area fractions 0.1 (☝), 0.2 (Δ), 0.3 (O), 0.4 (◊), 0.5 (∇) and 0.6 (▹). The lubrication theory result (10) is shown as a solid line. The scaling result (12) is shown as a dashed line. The aspect ratio *A*=0.2.

Fig. 5. Length scale ratio *L/a*, defined by (11), for flow of a non-Newtonian fluid through a rectangular array of elliptical cylinders along the major (open symbols) and minor (filled symbols) axis for different area fractions. The aspect ratio *A*=0.2. Symbols as in Fig. 4.

Fig. 6. Length scale ratio $\mathcal{L}/L_g$ as a function of solid area fraction for flow along the major axis (☝) and minor axis (Δ) of a non-Newtonian fluid through a rectangular array of elliptical cylinders. The aspect ratio *A*=0.2.

Fig. 7. Apparent permeability coefficients K_1 (open symbols) and K_2 (filled symbols) as a function of θ_F for an applied drag force at an angle θ_F with the major axis of the array for a power-law fluid with *n*=0.7 and ϕ=0.1 (☝), 0.3 (Δ) and 0.5 (O). The aspect ratio is 0.2.

Fig. 8. Angle of average velocity, θ, versus the stretched angle of the applied force, θ_F, for off-axis flows of a power-law fluid (*n*=0.7) at solid area fractions of 0.1 (☝), 0.3 (Δ) and 0.5 (O). The solid line indicates Eq. (13). The aspect ratio is 0.2.

Fig. 9. Contour plot of the permeability coefficient *K* for on-axis flow as a function of solid area fraction and aspect ratio of the cylinders for Newtonian fluids. For cylinders of aspect ratio *A*, K_1=*K*(A) and K_2=*K*(1/*A*). The contour values refer to ln(*K*).

Fig. 10. Velocity variance components R_{11}^{1} (open symbols) and R_{22}^{2} (filled symbols) in the main flow direction for on-axis flows as functions of power-law exponent, n, for solid area fractions 0.1 (✐), 0.2 (Δ), 0.3 (O), 0.4 (◊), 0.5 (∇) and 0.6 (▹). The solid line shows the lubrication result (15) for R_{11}^{1} (and $R_{22}{}^{2}$) at ϕ=0.6.

Fig. 11. Velocity variance components R_{22}^{1} (open symbols) and R_{11}^{2} (filled symbols) perpendicular to the flow direction for on-axis flows as a function of power-law exponent, n, for solid area fractions 0.1 (✐), 0.2 (Δ), 0.3 (O), 0.4 (◊), 0.5 (∇) and 0.6 (▹).

Fig. 12. Contour plot of the principal velocity variance component R for on-axis flows of Newtonian fluids, as a function of the solid area fraction and aspect ratio of the cylinders. For cylinders of aspect ratio A, $R_{11}^{1}=R(A)$ and $R_{22}^{2}=R(1/A)$. The contour values refer to R.

Fig. 13. Velocity variance components R_{11} and R_{22} for off-axis flow as functions of the stretched angle of the drag force, θ_F and solid area fraction, ϕ for Newtonian fluid (open symbols) and power-law fluid (n=0.7; solid symbols). R_{11}: (✐), ϕ=0.1; (Δ), ϕ=0.3; (O), ϕ=0.5. R_{22}: (◊), ϕ=0.1; (∇), ϕ=0.3; (▹), ϕ=0.5.