Interfacial instability of turbulent two-phase stratified flow: Pressure-driven flow and non-Newtonian layers

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We derive a flat-interface model to describe the flow of two horizontal, stably stratified fluids, where the bottom layer exhibits non-Newtonian rheology. The model takes into account the yield stress and power-law nature of the bottom fluid. In the light of the large viscosity contrast assumed to exist across the fluid interface, and for large pressure drops in the streamwise direction, the possibility for the upper Newtonian layer to display fully-developed turbulence must be considered, and is described in our model. We develop a linear-stability analysis to predict the conditions under which the flat-interface state becomes unstable, and pay particular attention to characterizing the influence of the non-Newtonian rheology on the instability. Increasing the yield stress (up to the point where unyielded regions form in the bottom layer) is destabilizing; increasing the flow index, while bringing a broader spectrum of modes into play, is stabilizing. In addition, a second mode of instability is found, which depends on conditions in the bottom layer. For shear-thinning fluids, this second mode becomes more unstable, and yet more bottom-layer modes can become unstable for a suitable reduction in the flow index. One further difference between the Newtonian and non-Newtonian cases is the development of unyielded regions in the bottom layer, as the linear wave on the interface grows in time. These unyielded regions form in the trough of the wave, and can be observed in the linear analysis for a suitable parameter choice.

Keywords: Herschel-Bulkley fluids; Turbulence; Interfaces; Instability

I. INTRODUCTION

We investigate the linear instability of two-layer pressure-driven flow in two dimensions as a model of conditions that commonly occur in industry: examples include the transport of waxy crude oils [1], and the removal of a layer of a non-Newtonian fluid by a Newtonian fluid (e.g. in the cleaning of surfaces, see [2]). Indeed, at present, it seems unclear whether interfacial waves have sufficient time to grow during cleaning operations, and in this context, it is important to formulate a predictive model for the growth rate of these waves. Previous work on this subject (reviewed briefly below) has been primarily for laminar flow conditions; studies on turbulent flows shearing past a viscous film has, to our knowledge, been confined to Newtonian fluids. In this paper we consider two-layer flows wherein the flow of the less viscous fluid is turbulent, and the more viscous fluid is a Herschel–Bulkley fluid and its motion is assumed to be laminar.

Previous work on the laminar Newtonian two-layer problem (cf. [3]) has shown the dominance of the so-called Yih-mode (or interfacial mode, see [4]), which is driven by the jump in

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viscosity (and hence shear rate) across the interface that leads to transfer of energy from the base state into the disturbance (cf. [5]). Only at sufficiently large Reynolds number values is a shear (or internal) mode observed, which is driven by energy transfer due to work done by the shear flow in the less-viscous layer away from the interface, and has a growth rate that is much lower than that of the interfacial mode.

Similar mechanisms have been found to dominate the instability of Newtonian films sheared by a turbulent flow, but the shear mode is dominated here by energy transfer in the liquid film (cf. [6]): again, an interfacial instability mechanism is dominant with the exception of small regions in parameter space. Incidently, the dominance of interfacial instability shows the importance of the inclusion of a viscous sublayer in the turbulent base state velocity profile near the interface. Ó Náraigh *et al.* [7, 8] found a transition to a Miles-type instability [9] only for very high Reynolds-number flows past liquid films, and for relatively thick films (i.e., fast waves). The study of Ó Náraigh *et al.* [7] showed that inclusion of perturbation turbulent stress terms in the analysis (with rapid distortion theory) has little effect on the prediction of the growth rate and wave speed, with the exception of fast waves. The flow pattern of the disturbance is affected by this, however.

In the present paper we investigate the corresponding problem for a liquid film consisting of a Herschel–Bulkley fluid. This problem has, to our knowledge, only been investigated for laminar conditions, and even then, only well away from a critical condition for the onset of unvielded regions to form [10]. The paper is organized as follows. In Sec. II we develop a basestate model that takes account of turbulence in the upper layer and non-Newtonian rheology in the lower layer. In Secs. III and IV we carry out a linear-stability analysis around the well-yielded base state (i.e. far from criticality), and demonstrate that the main differences between the Newtonian and inelastic non-Newtonian systems are primarily quantitative, the growth rate being affected by the difference in viscosity contrast across the interface (a similar result was found in the fully laminar case [10]). Next (Sec. VI), and in contrast to the study of Sahu et al. [10], we investigate flow conditions close to the critical value of the Bingham number, beyond which unvielded regions are formed. For flow conditions wherein an unyielded region is present in the base state, the system is superstable [11]; instead, we focus on base-state conditions wherein the Herschel–Bulkley layer is fully yielded, albeit barely so. We demonstrate that unstable waves will eventually grow to reach a point where unyielded regions may form. In this region of parameter space, the parallelism between Newtonian and non-Newtonian flows that held at lower Bingham numbers breaks down, and mode competition occurs, leading to the sidelining of the interfacial mode. Finally, in Sec. VII we present our conclusions.

II. THE FLAT-INTERFACE MODEL AND ITS PROPERTIES

In this section we derive a base state appropriate for a two-layer system in a channel, described schematically in Fig. 1. [LON: This idealised scenario is motivated by the practical problem of removal of viscous soils in industrial plants during cleaning and product turnover operations. Standard practice is to displace the more viscous product fluid (which initially fills entire pipelines) by water, resulting in viscous films being left behind on pipe walls by a finger of water. These films are sheared by the continued water flow, and we expect the resulting film displacement to be strongly influenced by the evolution of waves on the film surface.]

[LON: The base state we consider is due to Reynolds averaging of the turbulent system,



FIG. 1: A schematic diagram of the base flow. In this work, the bottom layer is laminar with non-Newtonian rheology, while the top layer exhibits fully-developed turbulence, described here by a Reynolds-averaged velocity profile. A pressure gradient in the *x*-direction drives the flow.

and represents an equilibrium state, in the sense that the average velocities are independent of time, and the average interfacial height that demarcates the phases is flat. LON: This approach, replacing instantaneous fluctuating fields with an averaged profile, is appropriate when the turbulent eddies turn over very slowly compared with the growth time of the waves, which is certainly the case here [12]. The correctness of this approach is further confirmed by the accurate predictions it gives for the critical Reynolds number of interfacial instability in the turbulent Newtonian case [6, 8, 13]. The bottom layer is a laminar liquid layer with non-Newtonian rheology, while the top layer is turbulent and fully-developed. A pressure gradient is applied along the channel. The mean profile of the system is a uni-directional flow in the horizontal, x-direction. In the top layer, near the interface (z = 0) and the wall (z = h), the flow profile is linear, and the viscous scale exceeds the characteristic length scale of the turbulence [14, 15]. In the top-layer core, the flow possesses a logarithmic profile [14, 15]. We also assume that the interface is smooth. The growth rate of the wave amplitude depends sensitively on the choice of mean flow and it is necessary to derive a mean flow-profile that incorporates the characteristics of the flow observed in experiments. In this section, we therefore generalize the model of O Náraigh et al. [8] which coupled laminar bottom-layer and turbulent top-layer flows to take into account the possibility of having a non-Newtonian bottom layer.

A. The mean-flow profile

The bottom layer: In this section we focus on the case where the base flow in the non-Newtonian layer is fully yielded. (We discuss other possible flow configurations in Sec. II C.) Thus, the velocity is non-zero everywhere except at the bottom wall (no slip), and the profile is determined from a balance between viscous and pressure forces:

$$\frac{\partial}{\partial z} \left(\mu \frac{\partial U_B}{\partial z} \right) - \frac{\partial p}{\partial x} = 0, \tag{1}$$

where U_B denotes the liquid mean flow velocity and the pressure gradient $\partial p/\partial x$, contains partial derivatives because hydrostatic balance is imposed in the vertical direction, $-\partial p_j/\partial z = \rho_j g$, in which j = B, T labels the phase. For flow from left to right, we have $\partial p/\partial x < 0$. The viscosity μ is non-constant in a non-Newtonian fluid and is constituted as

$$\mu_B = k \Pi^{n-1} + \tau_0 \Pi^{-1} \tag{2}$$

where Π is the second invariant of the rate-of-strain tensor,

$$\Pi = \sqrt{2E_{ij}E^{ij}}, \qquad E_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$
(3)

For fully-yielded flows, the division by zero implied by Eq. (2) does not occur, since the rate of strain is non-zero and positive. Thus, $\partial U_B/\partial z$ is positive throughout the liquid film, and for uni-directionl shear flow $\boldsymbol{U} = (U(z), 0)$, we obtain the relation $\Pi = |\partial U_B/\partial z| = \partial U_B/\partial z$, where the last equation holds for a fluid that has no unyielded regions. We integrate Eq. (1) twice and obtain the result

$$U_B = c_3 - \frac{n}{n+1} \frac{1}{|\partial p/\partial x|} \frac{1}{k^{1/n}} \left(-\left| \frac{\partial p}{\partial x} \right| z - c_2 - \tau_0 \right)^{1+1/n}, \tag{4}$$

where c_2 and c_3 are constants of integration that we determine. The shear stress at the interface is a constant, and given by

$$\tau_{\rm i} = \mu \frac{\partial U_B}{\partial z} \Big|_{z=0}.$$
(5)

For fully-yielded systems, where the interfacial shear stress is sufficient to overcome the yield stress and precipitate flow in the non-Newtonian layer, we have the condition $\tau_i > 0$. Thus, $\mu \left(\partial U_B / \partial z \right)_{z=0}$ is simply $k \left(\partial U_B / \partial z \right)_{z=0}^n + \tau_0$, and Eq. (5) becomes

$$k\left(\frac{\partial U_B}{\partial z}\right)_{z=0}^n + \tau_0 = \tau_{\rm i} > 0,$$

where the strict inequality holds for a fully yielded fluid. The gives the condition $\tau_0 < \tau_i$, that is, the stress applied to the interface must exceed the yield stress. Next, we determine the constants of integration. From Eq. (4), $(\partial U_B/\partial z)_{z=0}^n$ is simply $(c_2 + \tau_0)/k$, and hence $c_2 = -\tau_i$. Furthermore, applying the no-slip condition at the channel bottom $z = -d_B$ fixes c_3 , and we have the velocity profile

$$U_B = \frac{n}{n+1} \frac{1}{k^{1/n}} \frac{1}{|\partial p/\partial x|} \left[\left(d_B \left| \frac{\partial p}{\partial x} \right| + \tau_i - \tau_0 \right)^{1+1/n} - \left(- \left| \frac{\partial p}{\partial x} \right| z + \tau_i - \tau_0 \right)^{1+1/n} \right]$$
(6)

Next, we non-dimensionalize this equation using the upper-layer thickness h, the pressure $h|\partial p/\partial x|$, and the upper-layer superficial velocity $\rho_T U_0^2 = h|\partial p/\partial x|$:

$$\tilde{U}_B = \frac{n}{n+1} \frac{Re_0}{m} \left[\left(\delta + \frac{Re_*^2}{Re_0^2} - Bn \right)^{1+1/n} - \left(-z + \frac{Re_*^2}{Re_0^2} - Bn \right)^{1+1/n} \right]$$
(7)

where we have introduced the following important non-dimensional groups:

- 1. The viscosity contrast $m = \mu_{\rm e}/\mu_T$, where $\mu_{\rm e}$ is the effective liquid viscosity $\mu_{\rm e} = h |\partial p/\partial x| [k/(h|\partial p/\partial x|)]^{1/n}$;
- 2. The Reynolds number $Re_0 = \rho_T h U_0 / \mu_T$;
- 3. The friction Reynolds number $Re_* = (\rho_T h/\mu_T) \sqrt{\tau_i/\rho_T} = \rho_T h U_{*i}/\mu_T$;
- 4. The Bingham number $Bn = \tau_0 / (h |\partial p / \partial x|);$
- 5. The aspect ratio $\delta = d_B/h$.

Note that Eq. (7) reduces to the following form for the Newtonian case (Bn, n) = (0, 1):

$$\tilde{U}_B = \frac{Re_0}{2m} \left[\left(\delta + \frac{Re_*^2}{Re_0^2} \right)^2 - \left(-\tilde{z} + \frac{Re_*^2}{Re_0^2} \right)^2 \right], \qquad m = k/\mu_T,$$

for which $\left(\partial \tilde{U}_B/\partial \tilde{z}\right)_{z=0} = Re_*^2/(mRe_0)$. The friction Reynolds number is as yet undetermined, although it is available as a function of the control parameter Re_0 by careful consideration of the flow in the upper layer, to which we now turn.

The top layer: The RANS equation in the top layer is

$$\mu_T \frac{\partial U_T}{\partial z} + \tau_{\rm TSS} = \tau_{\rm i} + \frac{\partial p}{\partial x} z,\tag{8}$$

where $\tau_{TSS} = -\rho \langle u'w' \rangle$ is the turbulent shear stress due to the averaged effect of the turbulent fluctuating velocities. In general and *a priori*, there is no way of constituting this stress contribution in terms of averaged velocities and pressures, and a closure must be sought. We use the closure described by Ó Náraigh *et al.* [8], which was shown to give rise to a base state and linear-stability predictions that agree well with direct numerical simulations and experiments; it is a small step to generalize it here to the non-Newtonian lower-layer rheology. Thus, we constitute the turbulent shear stress as follows:

$$\tau_{\rm TSS} = \kappa \rho_T h U_{\rm *w} G\left(\tilde{z}\right) \psi_{\rm i}\left(\tilde{z}\right) \psi_{\rm w} \left(1 - \tilde{z}\right) \frac{\partial U_T}{\partial z},\tag{9}$$

where κ is the Von Kárman constant, U_{*w} is the friction velocity at the upper wall, and is related to the friction velocity at the interface through the relation $U_{*i}^2/U_{*w}^2 = R$, with the ratio R constituted below. The interpolating function G has the following form [16]:

$$G(s) = s(1-s) \underbrace{\left[\frac{s^3 + |R|^{5/2}(1-s)^3}{R^2(1-s)^2 + Rs(1-s) + s^2}\right]}_{=\mathcal{V}(s)} \qquad 0 \le s \le 1,$$
(10)

while we use a Van-Driest type of formalism [15] for the wall functions ψ_i and ψ_w :

$$\psi_{i}(s) = 1 - e^{-s^{p}/A_{i}}, \qquad \psi_{w}(1-s) = 1 - e^{-(1-s)^{n}/A_{w}}.$$
 (11)

The input parameters p, A_i , and A_w are chosen such that p = 2, and such that the interfacial and wall laminar sublayers are five interfacial (wall) units in extent. [LON: Moreover,



FIG. 2: Solutions to the base-state model with a turbulent upper layer, with comparisons to the equivalent laminar flow profile, r = 2, m = 10, and $\delta = 0.2$. The turbulent Reynolds number is $Re_0 = 1000$, while the laminar Reynolds number is chosen such that the laminar and turbulent flow rates match. Figures (a) and (b) show the solution for Bn = 0 and various *n*-values in the entire domain and in the bottom layer respectively; a comparison with the laminar case at n = 1 is shown in (a). Figures (c) and (d) show the solution for Bn = 0.3 and various *n*-values.

implicit in our choice of G-function is the assumption that interfacial roughness is small, in the sense described by that is, there is no surface roughness, in the sense described by Miles [9], Akai and co-workers [17, 18], and Biberg [16]. This assumption is valid for low-intensity turbulence, and alters the results only qualitatively at higher intensities.]

Next, we solve for the velocity profile up to quadrature. Substituting Eq. (9) into Eq. (8)

gives the formula

$$U_{T}(z) = U_{T}(0) + \tau_{i}h \int_{0}^{z/h} \frac{\left(1 + \frac{h}{\tau_{i}}\frac{\partial p}{\partial x}s\right)ds}{\mu_{T} + \kappa\rho_{T}hU_{*w}G(s)\psi_{i}(s)\psi_{w}(1-s)},$$

$$= U_{T}(0) + \tau_{i}h \int_{0}^{z/h} \frac{\left(1 + \frac{h}{\tau_{i}}\frac{\partial p}{\partial x}s\right)ds}{\mu_{T} + \frac{\kappa\rho_{T}hU_{*i}}{\sqrt{|R|}}G(s)\psi_{i}(s)\psi_{w}(1-s)},$$
(12)

where $R = \tau_i / \tau_w$. Non-dimensionalizing and using the matching condition

$$\tilde{U}_T(0) = \tilde{U}_B(0) = \frac{n}{n+1} \frac{Re_0}{m} \left[\left(\delta + \frac{Re_*^2}{Re_0^2} - Bn \right)^{1+1/n} - \left(\frac{Re_*^2}{Re_0^2} - Bn \right)^{1+1/n} \right],$$

this is

$$\tilde{U}_{T}\left(\tilde{z}\right) = \frac{n}{n+1} \frac{Re_{0}}{m} \left[\left(\delta + \frac{Re_{*}^{2}}{Re_{0}^{2}} - Bn \right)^{1+1/n} - \left(\frac{Re_{*}^{2}}{Re_{0}^{2}} - Bn \right)^{1+1/n} \right] \\ + \frac{Re_{*}^{2}}{Re_{0}} \int_{0}^{\tilde{z}} \frac{\left(1 - \frac{Re_{0}^{2}}{Re_{*}^{2}} s \right) ds}{1 + \frac{\kappa Re_{*}}{\sqrt{|R|}} G\left(s \right) \psi_{i}\left(s \right) \psi_{w}\left(1 - s \right)}.$$
(13)

The ratio R can be obtained in closed form as follows. Since

$$\begin{aligned} \tau\left(z\right) &= \tau_{\rm i} + \frac{\partial p}{\partial x} z, \\ &= -\tau_{\rm w} + \frac{\partial p}{\partial x} \left(z - h\right), \end{aligned}$$

these formulas can be equated to give

$$\tau_{\rm i} = -\tau_{\rm w} - \frac{\partial p}{\partial x}h_{\rm s}$$

hence,

$$|R| = \left|1 - \left(\frac{Re_0}{Re_*}\right)^2\right|^{-1}$$

.

Finally, Re_* is determined as the zero of the function $\tilde{U}(1; Re_*) = 0$, or

$$\frac{n}{n+1} \frac{Re_0}{m} \left[\left(\delta + \frac{Re_*^2}{Re_0^2} - Bn \right)^{1+1/n} - \left(\frac{Re_*^2}{Re_0^2} - Bn \right)^{1+1/n} \right] \\ + \left\{ \frac{Re_*^2}{Re_0} \int_0^1 \frac{\left(1 - \frac{Re_0^2}{Re_*^2} s \right) ds}{1 + \frac{\kappa Re_*}{\sqrt{|R|}} G\left(s\right) \psi_{\rm i}\left(s\right) \psi_{\rm w}\left(1-s\right)} \right\}_{|R| = \left| 1 - \left(\frac{Re_0}{Re_*} \right)^2 \right|^{-1}} = 0.$$
(14)



FIG. 3: Solutions to the base-state model with a turbulent upper layer, where $Re_0 = 1000$, r = 2, m = 10, and $\delta = 0.2$. (a) Dependence of $U_{T,\max}/U_{B,\max}$ on Bingham number, with n = 1. The condition for the liquid layer to contain no unyielded regions is $Re_* \ge \sqrt{Bn}Re_0$; (b) dependence of $U_{T,\max}/U_{B,\max}$ on the flow index, with Bn = 0.

In summary, we have the following velocity profile in the base state:

$$\begin{split} \tilde{U}\left(\tilde{z}\right) &= \\ \frac{n}{n+1} \frac{Re_0}{m} \left[\left(\delta + \frac{Re_*^2}{Re_0^2} - Bn \right)^{1+1/n} - \left(-z + \frac{Re_*^2}{Re_0^2} - Bn \right)^{1+1/n} \right], \qquad -\delta \leq \tilde{z} \leq 0, \\ \tilde{U}\left(\tilde{z}\right) &= \frac{n}{n+1} \frac{Re_0}{m} \left[\left(\delta + \frac{Re_*^2}{Re_0^2} - Bn \right)^{1+1/n} - \left(\frac{Re_*^2}{Re_0^2} - Bn \right)^{1+1/n} \right] \\ &+ \frac{Re_*^2}{Re_0} \int_0^{\tilde{z}} \frac{\left(1 - \frac{Re_0^2}{Re_*^2} s \right) ds}{1 + \frac{\kappa Re_*}{\sqrt{|R|}} G\left(s \right) \psi\left(s \right) \psi\left(1 - s \right)}, 0 \leq \tilde{z} \leq 1. \end{split}$$
(15)

Solutions to this equation are shown in Fig. 2 for various flow indices and Bingham numbers. Also shown is a comparison with what the laminar profile would be if such a flow configuration could be sustained at the same flow rate: the turbulent and laminar profiles are substantially different. In Fig. 3, we demonstrate the dependence of the maximum velocity on the Bingham number and flow index, at a fixed Reynolds number. By re-arranging these data, and using the equation (15), we obtain the dependence of the characteristic Reynolds numbers

$$Re_{T,\max} = \frac{Re_0 U_{T,\max}}{U_0}, \qquad Re_B = \frac{Re_0 U_B(0)}{U_0} \frac{r\mu_e}{\delta m \mu_{B,\max}},$$

on the flow parameters n and Bn. The results of this re-arrangement are shown in Fig. 4, which demonstrates that even at small values of the viscosity and density contrasts (specifically, r = 2 and m = 10), it is possible to generate a large Reynolds-number contrast across the interface. This contrast is enhanced for Bingham and shear-thickening fluids, whose



FIG. 4: Solutions to the base-state model with a turbulent upper layer, where $Re_0 = 1000$, r = 2, m = 10, and $\delta = 0.2$. (a) Dependence of $Re_{T,\max}$ and $Re_{B,\max}$ on Bingham number, with n = 1. The parameter $Re_{B,\max}$ decreases with increasing Bn; (b) dependence of $Re_{T,\max}$ and $Re_{B,\max}$ on the flow index, with Bn = 0. The parameter $Re_{B,\max}$ decreases (relative to the Newtonian fluid) for shear-thickening fluids. These two results demonstrate possibility that even at small values of the density and viscosity contrasts, it is possible to generate a large Reynolds-number contrast across the interface, which points to the realism of modelling a laminar non-Newtonian layer sheared by a turbulent, Newtonian, upper layer.

viscosity is enhanced at larger Bn- and n-values respectively. This underscores the realism of the physical system described by Fig. 1, namely a non-Newtonian, laminar bottom layer sheared by a Newtonian, fully-developed, turbulent upper layer. In summary, the results of Figs. 2 and 4 suggest that we should expect significant differences between the stability properties of the laminar and turbulent systems with respect to a linear stability analysis, which is the subject of Sec. III

B. Accuracy of the base-state model

To our knowledge, the turbulent profiles described here have not been measured in experiments or by direct numerical simulation (DNS). These measurements have been made for fully Newtonian systems however, and in those situations, the model used here provides accurate predictions. In particular, Ó Náraigh *et al.* [8] have compared the fully Newtonian, laminar-turbulent base state with the DNS results of Solbakken and Andersson [19], and excellent agreement is obtained. They also verified that the single-phase, Newtonian, fully-turbulent version of the model agrees with experiments. Furthermore, a linear-stability analysis based on this fully Newtonian base state provides predictions for the critical Reynolds number of instability that are in excellent agreement with the experiments of both Craik [20], and Cohen and Hanratty [21]. Since the turbulent modelling affects only the upper layer, and since this layer is the same regardless of the rheology of the bottom layer, we are confident in the generalization we have performed in applying this turbulence model to the present case.

C. The critical Bingham number

In Sec. II A we investigated base-flow configurations for which the interfacial shear stress τ_i was sufficient to overcome the Bingham nature of the non-Newtonian layer and cause the liquid to flow throughout the channel. Specifically, we had the condition $Re_* > \sqrt{BnRe_0}$, suggesting a critical Bingham number $Bn_c = Re_0^2/Re_*^2$. In this section we investigate cases where unyielded domains form as a result of the finite yield stress $(Bn \ge Bn_c)$. To take account of the singularity in the viscosity $\mu \sim k\Pi^{n-1} + Bn/\Pi$ as the rate of strain II tends to zero, we introduce a regularization in the expression (2), valid at small strain rates, while for simplicity, we focus on fluids with power-law index n = 1. While the convergence of the regularized model to the singular model has been found to be problematic [22, 23], our application of this regularization is not to the linear-stability analysis itself, but is simply a means of giving a qualitative description of the formation of unyielded regions in the wavy state, given a fully-yielded base state; this is done in Sec. VI. Thus, the regularization introduced here can be regarded as a simple expediency, and casts no doubt on the results of the linear-stability calculations.

The regularized, so-called bi-viscosity Herschel–Bulkley model takes account of the singularity as $\Pi \rightarrow 0$ by the introduction of a regularized viscosity at low shear rates [22, 24–27]:

$$\mu = \begin{cases} \mu_0, & \Pi \le \Pi_0 \\ k\Pi^{n-1} + \tau_0 \Pi^{-1}, & \Pi \ge \Pi_0 \end{cases}.$$
 (16a)

Integrating the momentum balance equation and non-dimensionalizing in the usual way (with $\pi_0 = \prod_0 / (\rho_T U_0^2)$), we obtain the velocity profile

$$U_B = \begin{cases} U(0) + z \left(\pi_0 Re_0 - \frac{Re_0 h_0}{m} \frac{1}{1 + \frac{Bn}{\pi_0 m}} \right) - \frac{1}{1 + \frac{Bn}{\pi_0 m}} \frac{Re_0}{2m} z^2, & -h_0 < z < 0, \\ \frac{1}{m} \left(\frac{Re_*^2}{Re_0^2} - Bn \right) (z + \delta) - \frac{Re_0}{2m} (z^2 - \delta^2), & -\delta < z < -h_0, \end{cases}$$
(16b)

and the stress

$$\tau_B = \begin{cases} \frac{1}{Re_0} \left(m + \frac{Bn}{\pi_0} \right) \frac{\mathrm{d}U_B}{\mathrm{d}z}, & -h_0 < z < 0, \\ \frac{1}{Re_0} \frac{\mathrm{d}U_B}{\mathrm{d}z} + Bn, & -\delta < z < -h_0. \end{cases}$$
(16c)

Note that the second derivative of the stress is continuous, while the second derivative of U_B is not. The constant h_0 is determined by the stress at the interface and the pressure drop:

$$h_0 = m\pi_0 + Bn - \frac{Re_*^2}{Re_0^2}.$$
 (16d)

When $0 < h_0 < \delta$ there is a *yield surface* at $z = -h_0$, across which the viscosity changes character. The presence or otherwise of the yield surface gives rise to three distinct regimes, listed as follows and depicted schematically in Fig. 5.

1. Well-yielded flows: These are flows for which $h_0 < 0$, and thus the crossover into the regularized zone is never attained.



FIG. 5: Summary of scenarios given by solution of Eqs. (16).

- 2. Quasi-yielded flows: These are flows for which $0 < h_0 < \delta$. Then, the base state in the non-Newtonian layer consists of a plug flow near the interface (the 'unyielded region'), and a boundary near the solid wall $z = -\delta$ where the flow transitions smoothly to zero.
- 3. Unyielded flows: These are flows for which $h_0 > \delta$, and for which the viscosity attains its maximal value everywhere in the non-Newtonian flow domain. The magnitude of the velocity is very small everywhere.

Cases (2) and (3) can be treated within the framework of single-phase flow [11], and are not considered here. Thus, in Secs. III–V we focus on case (1). Of interest too is the parameter regime in which the critical Bingham number Bn_c is approached from below, which we call the *barely-yielded* regime. This is the focus of Sec. VI, where we demonstrate that linear waves that develop from a fully-yielded state can produce unyielded regions near the interface.

III. LINEAR-STABILITY ANALYSIS: INTERFACIAL MODE

In this section we outline a linear-stability analysis that determines the conditions under which the interface of the two-phase system becomes unstable. For viscous, parallel, incompressible flow, the stability analysis reduces to an eigenvalue problem in a single equation (the Orr–Sommerfeld equation). We generalize this approach to turbulent stratified flow by performing a stability analysis on the flat-interface Reynolds-averaged Navier–Stokes (RANS) equations. We introduce a numerical method for the solution of the eigenvalue problem produced by our Orr–Sommerfeld analysis. We verify this numerical method against results for laminar flow.

A. Orr–Sommerfeld analysis

The base state $U_j(z) = (U_j(z), 0), j = B, T$ is a parallel flow given by the model equations in Sec. II. Infinitesimal perturbations to the base state are described by the

linearized RANS equations:

$$r_{j}\left(\frac{\partial \boldsymbol{u}_{j}}{\partial t} + \boldsymbol{U}_{j} \cdot \nabla \boldsymbol{u}_{j} + \boldsymbol{u}_{j} \cdot \nabla \boldsymbol{U}_{j}\right) = \nabla \cdot \mathsf{T}_{j} + \nabla \cdot \mathsf{T}_{\mathrm{Rey},j} - r_{j}\left(\frac{gh}{U_{0}^{2}}\right)\hat{\boldsymbol{z}}$$
(17a)

$$\nabla \cdot \boldsymbol{u}_j = 0, \tag{17b}$$

where $r_B = r$, and $r_T = 1$. In the non-Newtonian layer, the turbulent component of the stress tensor is aassumed to be zero, while the viscous component of the stress tensor has the form

$$\mathsf{T}_{xx,B} = -p + \frac{2\mu_0}{Re_0} \frac{\partial u}{\partial x}, \qquad \mathsf{T}_{zz,B} = -p + \frac{2\mu_0}{Re_0} \frac{\partial w}{\partial z},$$
$$\mathsf{T}_{xz,B} = \frac{\mu_0}{Re_0} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \frac{\beta_0}{Re_0} \frac{\mathrm{d}U_B}{\mathrm{d}z}\pi,$$

where

$$\mu_0 = m \left(\frac{\mathrm{d}U_B}{\mathrm{d}z}\right)^{n-1} + Bn \left(\frac{\mathrm{d}U_B}{\mathrm{d}z}\right)^{-1},$$
$$\beta_0 = (n-1) m \left(\frac{\mathrm{d}U_B}{\mathrm{d}z}\right)^{n-2} - Bn \left(\frac{\mathrm{d}U_B}{\mathrm{d}z}\right)^{-2},$$

and where

$$\pi = \sqrt{\frac{1}{2} \sum_{\alpha,\beta=1}^{2} \left[\frac{\partial \left(U_{\alpha} + u_{\alpha} \right)}{\partial x_{\beta}} + \frac{\partial \left(U_{\beta} + u_{\beta} \right)}{\partial x_{\alpha}} \right] \left[\frac{\partial \left(U_{\alpha} + u_{\alpha} \right)}{\partial x_{\beta}} + \frac{\partial \left(U_{\beta} + u_{\beta} \right)}{\partial x_{\alpha}} \right] - \frac{\mathrm{d}U_{B}}{\mathrm{d}z}} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \left(\mathrm{D}^{2} + \alpha^{2} \right) \phi_{B}, \qquad U_{\alpha} = U_{B} \delta_{\alpha 1}, \quad (18)$$

is the perturbation to the second invariant of the rate-of-strain tensor. Similarly, in the Newtonian top layer, the viscous component of the stress tensor is expressed as

$$\mathsf{T}_{xx,T} = -p + \frac{2}{Re_0} \frac{\partial u}{\partial x}, \qquad \mathsf{T}_{zz,T} = -p + \frac{2}{Re_0} \frac{\partial w}{\partial z}, \qquad \mathsf{T}_{xz,T} = \frac{1}{Re_0} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right),$$

but here the turbulent component of the stress tensor is non-trivial and has the form

$$\mathsf{T}_{\mathrm{Rey},j} = -r_j \langle \boldsymbol{u}_j' \otimes \boldsymbol{u}_j' \rangle - \mathsf{T}_{\mathrm{Rey},j}^{(0)},$$

where \boldsymbol{u}' is the pre-averaged, fully turbulent, fluctuational velocity field, and $\mathsf{T}^{(0)}_{\operatorname{Rey},j}$ is the Reynolds stress in the flat-interface state, modelled in Sec. II. For simplicity, we ignore contributions from this component of the tensor. This is legitimate, since the effects of the turbulence come almost entirely from the base state, especially for viscosity-contrast instabilities; see [8]. Next, we re-express these equations in terms of the streamfunction

$$(u_j, w_j) = (\partial \Phi_j / \partial z, -\partial \Phi_j / \partial x)$$

and use the Fourier decomposition $\Phi_j = \phi_j(z) e^{i\alpha(x-ct)}$. By rewriting the wave speed in terms of its real and imaginary parts, $c = c_r + ic_i$, the exponential component of the solution can be re-expressed as $e^{\alpha c_i t} e^{i\alpha(x-c_r t)}$. Thus, positivity of c_i indicates a growing instability;

the real part of the constant $\lambda = -i\alpha (c_r + ic_i)$ is the growth rate, and this depends on the problem parameters, that is,

$$\lambda = \lambda \left(\alpha, r, m, \delta, Re_0, G, \Gamma \right).$$

Here G and Γ represent a Bond number and an inverse capillary number respectively, defined below in (22) and (23). Using the streamfunction representation of the velocity field, the RANS equations (17) reduce to a pair of coupled Orr–Sommerfeld-type equations,

$$i\alpha r Re_{0} \left\{ \left[U_{B}(z) - c \right] \left(D^{2} - \alpha^{2} \right) \phi_{B} - \phi_{B} \frac{d^{2} U_{B}}{dz^{2}} \right\} = \left(D^{2} - \alpha^{2} \right)^{2} \phi_{B} + a_{0}(z) \phi_{B} + a_{1}(z) D\phi_{B} + a_{2}(z) D^{2} \phi_{B} + a_{3}(z) D^{3} \phi_{B} + a_{4}(z) D^{4} \phi_{B} \qquad -\delta \leq z < 0,$$
(19a)

$$i\alpha Re_0 \left\{ \left[U_T(z) - c \right] \left(D^2 - \alpha^2 \right) \phi_T - \phi_T \frac{d^2 U_T}{dz^2} \right\} = \left(D^2 - \alpha^2 \right)^2 \phi_T, \qquad 0 \le z \le 1, \quad (19b)$$

where we have taken D to mean d/dz, and where

$$\begin{aligned} a_{0}(z) &= \alpha^{2} \frac{d^{2} \mu_{0}}{dz^{2}} + \alpha^{2} \left[\beta_{0} \left(\frac{d^{3} U_{B}}{dz^{3}} + \alpha^{2} \frac{dU_{B}}{dz} \right) + 2 \frac{d\beta_{0}}{dz} \frac{d^{2} U_{B}}{dz^{2}} + \frac{d^{2} \beta_{0}}{dz^{2}} \frac{dU_{B}}{dz} \right], \\ a_{1}(z) &= -2\alpha^{2} \frac{d\mu_{0}}{dz} + \alpha^{2} \left(2\beta_{0} \frac{d^{2} U_{B}}{dz^{2}} + 2 \frac{d\beta_{0}}{dz} \frac{dU_{B}}{dz} \right), \\ a_{2}(z) &= \frac{d^{2} \mu_{0}}{dz^{2}} + \beta_{0} \left(\frac{d^{3} U_{B}}{dz^{3}} + \alpha^{2} \frac{dU_{B}}{dz} \right) + 2 \frac{d\beta_{0}}{dz} \frac{d^{2} U_{B}}{dz^{2}} + \frac{d^{2} \beta_{0}}{dz^{2}} \frac{dU_{B}}{dz} + \alpha^{2} \beta_{0} \frac{dU_{B}}{dz}, \\ a_{3}(z) &= 2 \frac{d\mu_{0}}{dz} + 2\beta_{0} \frac{d^{2} U_{B}}{dz^{2}} + 2 \frac{d\beta_{0}}{dz} \frac{dU_{B}}{dz}, \\ a_{4}(z) &= \beta_{0} \frac{dU_{B}}{dz}. \end{aligned}$$

Since the base state has two distinct domains (top and bottom layers), it is appropriate to solve the Orr–Sommerfeld equation in each domain, and to apply matching conditions at the domain boundary. This boundary or interface is at z = 0 + [perturbations]; here the perturbation velocities must be continuous, while the jump conditions on the stress tensor are standard. Linearizing these conditions on to the flat interfaces z = 0, we obtain the following conditions:

$$\phi_T = \phi_B, \tag{20a}$$

$$D\phi_T = D\phi_B + \frac{\phi_B}{c - U_B} \left(\frac{dU_B}{dz} - \frac{dU_{T,}}{dz} \right),$$
(20b)

$$\left(\mathrm{D}^{2}+\alpha^{2}\right)\phi_{T}=\left(\mu_{0}+\beta_{0}\frac{\mathrm{d}U_{B}}{\mathrm{d}z}\right)\left(\mathrm{D}^{2}+\alpha^{2}\right)\phi_{B},$$
(20c)

$$i\alpha Re_0 r \left[(c - U_B) D\phi_B + \frac{\mathrm{d}U_B}{\mathrm{d}z} \phi_T \right] + \mu_0 \left(D^3 \phi_B - 3\alpha^2 D\phi_B \right) + b_0 (z) \phi_B + b_2 (z) D^2 \phi_B + b_3 (z) D^3 \phi_B$$
$$= i\alpha Re_0 \left[(c - U_T) D\phi_T + \frac{\mathrm{d}U_T}{\mathrm{d}z} \phi_T \right] + \left(D^3 \phi_T - 3\alpha^2 D\phi_T \right) + i\alpha \left(\Gamma \alpha^2 + G \right) \frac{\phi_B}{c - U_B}, \quad (20d)$$

where

$$b_{0}(z) = \alpha^{2} \left[\frac{\mathrm{d}\mu_{0}}{\mathrm{d}z} + \beta_{0} \frac{\mathrm{d}^{2}U_{B}}{\mathrm{d}z^{2}} + \frac{\mathrm{d}\beta_{0}}{\mathrm{d}z} \frac{\mathrm{d}U_{B}}{\mathrm{d}z} \right]$$

$$b_{1}(z) = \alpha^{2}\beta_{0} \frac{\mathrm{d}U_{B}}{\mathrm{d}z},$$

$$b_{2}(z) = \frac{\mathrm{d}\mu_{0}}{\mathrm{d}z} + \beta_{0} \frac{\mathrm{d}^{2}U_{B}}{\mathrm{d}z^{2}} + \frac{\mathrm{d}\beta_{0}}{\mathrm{d}z} \frac{\mathrm{d}U_{B}}{\mathrm{d}z},$$

$$b_{3}(z) = \beta_{0} \frac{\mathrm{d}U_{B}}{\mathrm{d}z}.$$

These interfacial conditions are non-standard and are derived in Appendix B. Finally, to close the system of equations and to form an eigenvalue problem, we impose the no-slip conditions at the boundaries,

$$\phi_B\left(-\delta\right) = \phi'_B\left(-\delta\right) = 0,\tag{21a}$$

$$\phi_T(1) = \phi'_T(1) = 0. \tag{21b}$$

Note that gravity and surface tension have been introduced to the problem through the interfacial condition (20d): here G is the inverse Froude number

$$G = \frac{gh^2}{\rho_G U_0 \mu_T} \left(\rho_B - \rho_T\right),\tag{22}$$

and Γ is the inverse Weber number,

$$\Gamma = \frac{\sigma}{U_0 \mu_T},\tag{23}$$

where we have made use of the gravitational and surface tension constants g and σ . Note that these parameters contain explicit Reynolds-number dependence, through the turbulent velocity U_0 : this is discussed further at the start of Sec. IV. Having now outlined the model problem, we discuss a numerical method for its solution.

B. Numerical method and validation

Following standard practice, we build the boundary and interfacial conditions into the definition of our eigenvalue problem (19)–(21), which can then be written in operator form as

$$\mathsf{L}\phi = \lambda \mathsf{M}\phi, \qquad \lambda = -i\alpha c. \tag{24}$$

Next, we write a trial solution $\phi_B = \sum_{j=0}^{n_B} a_j T_j(z)$ and $\phi_T = \sum_{j=0}^{n_T} b_j T_j(z)$, where $T_j(z)$ is the j^{th} Chebyshev polynomial. We substitute these trial solutions into the eigenvalue problem (24) and evaluate the resulting equation at $n_B + n_T + 2$ interior, boundary and interfacial points. The resulting equation set is readily solved for λ . Choosing optimal values of n_B and n_T guarantees convergence of the numerical scheme (typical values are $n_B = 40, n_T = 80$, for δ -values between 0.05 and 0.5). We verify our numerical techniques with reference to the work of Sahu *et al.* [10] for *laminar* upper layers in Fig. 6. To compare with the existing results, we have taken $Re_0 = 24.40, Bn = 0.1679, m = 10, r = 1.1$,



FIG. 6: Validation of numerical method: comparison with the work of Sahu *et al.* for laminar flow, with $Re_0 = 24.40$, Bn = 0.1679, m = 10, r = 1.1, $\delta = 1$, $G = (1000/Re_0)(1 + \delta)^{-3}$, and $S = (100/Re_0)(1 + \delta)^{-1}$.

 $\delta = 1, G = (1000/Re_0) (1 + \delta)^{-3}$, and $S = (100/Re_0) (1 + \delta)^{-1}$, which enables a comparison with Fig. 4 of Sahu *et al.*. The existing method for laminar flow and the present method, when applied to laminar flow, give perfect agreement. However, the work of Sahu *et al.* omitted the contribution $\beta_0 U'_B(0)$ to the effective liquid viscosity at z = 0 (but see the corrigendum [28]). Including this contribution reduces the growth rate in this instance, since $\beta_0 (0) = -Bn [U'_B(0)]^{-2} < 0$ for n = 1, thereby reducing the viscosity contrast at the interface. This result highlights the important role that viscosity plays in the interfacial instability. Since the effective viscosity at the interface is a function of the Bingham number and the flow index, we proceed to a careful characterization of this dependence for the case of turbulent flow in the Newtonian layer.

IV. LINEAR-STABILITY ANALYSIS: RESULTS

Based on the method developed in Sec. III, we carry out a linear-stability analysis of the flat-interface state. In particular, we examine the effects of varying the Bingham number and the flow index, effects which are absent from the purely Newtonian case, which has been considered elsewhere [8]. To carry out the stability analysis, we must first of all find a reasonable estimate for the gravity- and surface-tension numbers, obtained in Eqs. (22) and (23) respectively. We estimate the gravity number by re-arranging it as

$$G = \frac{gh}{(\mu_T / (\rho_T h))^2} \frac{r - 1}{Re_0^2}.$$
 (25)

Based on $\mu_T = 10^{-3}$ Pa s, $\rho_G = 10^3$ kg m⁻³, and h = 0.05 m, the prefactor is 1.225×10^9 . The value of the interfacial tension coefficient Γ between the non-Newtonian soil layer and the upper fluid layer will vary depending on the substances under consideration. Precise values are indeed known for various combinations (see, for example, the work of Kristensen *et al.* for dairy products [29]). Here we set it to 10, which ensures that gravity dominates



FIG. 7: Comparisons between the growth rates for the turbulent and laminar models, Bn = 0.25, r = 2, m = 10, and $\delta = 0.2$. (a) Turbulent and laminar flow for the same value of the Reynolds number, $Re_0 = 1000$; (b) Turbulent and laminar flow at the same flow rate, $\int_{-\delta}^{1} U(z) dz = 14.13$, and $Re_0 = 1000$ for the turbulent case.

over surface tension except at very small wavelengths $\ell/h \leq 2\pi \sqrt{\Gamma/G}$. This involves no loss of generality, and we have verified (but do not show here) the qualitative similarity between the results obtained in the present case by varying this parameter, and the results obtained elsewhere, where the effects of varying the surface tension are studied for Newtonian [6, 8] and non-Newtonian [10] layers sheared by turbulent and laminar flow respectively.

The first aspect of our stability analysis is a comparison with the results for laminar flow. We compare the growth rate for the turbulent profile derived in Sec. II with that of a Poiseuille profile. Such a comparison necessitates the maintenance of the unstable two-phase laminar profile at high Reynolds numbers, which may not be possible. The results of the first of these comparisons is shown in 7 (a), where the Reynolds number Re_0 is kept fixed. The maximum growth rate is larger in the turbulent case. Note, however, the existence of two competing modes in the laminar case, which vanish in the turbulent case. These marked differences between the two models underscore the importance of studying the turbulent case. The second comparative result is shown in 7 (b), where the flow rate is held fixed. Again, the turbulent case is more unstable, and the difference is even more pronounced than in case (a).

Next, the results of the stability analysis for various Bn-values is shown in Fig. 8, where n = 1, $Re_0 = 1000$, r = 2, m = 10, and G and Γ are defined above (see Eq. (25)). This parameter choice corresponds to the flow of a turbulent Newtonian layer over a denser, more viscous, laminar Bingham fluid. In each case considered, the flow is unstable to a band of wavenumbers in the range $0 < \alpha < \alpha_c$, where α_c is the 'cutoff' wavenumber beyond which gravity and surface tension stabilize the flow. In Fig. 8 (b) the aspect ratio δ is set to 0.2. Increasing the Bingham number is seen to destabilize the flow: the cutoff wavenumber is shifted outwards, and the maximum growth rate is shifted upwards. This trend persists upon increasing the aspect ratio ($\delta = 0.5$ in Fig. 8 (c)). Increasing the aspect ratio further is beyond the capability of the model, since such an increase would lead to a large liquid Reynolds number ($Re_B = 1275$ for $\delta = 1$), which would call into question the assumption



FIG. 8: The effect of varying the Bingham number, Bn, on the linear stability of the base flow: (a) $\delta = 0.05$, (b), $\delta = 0.2$, and (c), $\delta = 0.5$. Here, n = 1, $Re_0 = 1000$, r = 2, m = 10, and G and Γ are defined according to the discussion in the introduction to this section. Increasing the Bingham number (at fixed pressure gradient $Re_0 = 1000$) is seen to be destabilizing: the cutoff wavenumber is shifted to higher values, and the maximum growth rate is increased. This effect becomes more prominent at larger δ -values.



FIG. 9: The effect of varying the flow index, n, on the linear stability of the base flow: (a) $\delta = 0.05$, (b), $\delta = 0.2$, and (c), $\delta = 0.5$. Here, Bn = 0, $Re_0 = 1000$, r = 2, m = 10, and G and Γ are defined according to the discussion in the introduction to this section. Increasing the flow index is stabilizing, since the maximum growth rate is shifted downwards. The dependence of the cutoff wavenumber on the flow index is non-monotonic: both an increase, and a decrease in n can increase the cutoff wavenumber, as evidenced in particular by the $\delta = 0.05$ curves.

that this layer is laminar. Next, we study the effects of varying the flow index in Fig. 9, at fixed Bn = 0. Increasing the flow index gives rise to a *reduction* in the maximum growth rate, for $\delta = 0.05$, 0.2, and $\delta = 0.5$. In contrast to the case where the Bingham number was varied at fixed n = 1, here the variation in the non-Newtonian parameter produces a change in the *shape* of the dispersion curve, rather than a simple overall upward or downward shift in the curve. Thus, increasing n beyond n = 1 decreases the maximum growth rate, but shifts the cutoff wavenumber to higher values, which produces instability for a wider band of disturbances. This dependence of the cutoff wavenumber on the flow index is highly non-monotonic however, as evidenced by a close inspection of Fig. 9 (a)–(c), and by Fig. 10,



FIG. 10: Dependence of the cutoff wavenumber on the flow index n for $\delta = 0.05$. Here Bn = 0, $Re_0 = 1000$, r = 2, m = 10, and G and Γ are defined according to the discussion in the introduction to this section. The dependence of the cutoff on the flow index is highly non-monotonic, with a minimum at $n \approx 0.7$.



FIG. 11: The effect of varying the Reynolds number, Re_0 , on the linear stability of the base flow, where, Bn = 0.25, r = 2, m = 10, and $\delta = 0.2$.

where the cutoff wavenumber as a function of n is shown, for $\delta = 0.05$. We also examine the effects of varying the Reynolds number in Fig. 11, where an increase in the Reynolds number is shown to be destabilizing, through an increase in both the maximum growth rate, and the cutoff wavenumber. Finally, we turn to a calculation that demonstrates the effects of varying m, r, and δ on the stability characteristics of the system.

The effect of varying the parameters m, r and δ on the stability properties of a two-layer system has been discussed in the literature for *Newtonian* fluids [8], and we expect the results to be qualitatively similar for the case considered here (non-Newtonian films sheared by turbulent upper layers). However, to develop a quantitative description, we provide a comparison between the Newtonian and the Bingham cases in Fig. 12; this figure also serves to demonstrate the destabilizing effects of the non-Newtonian rheology. Thus, in each case considered, the presence of the yield stress is destabilizing, in the sense that the maximum growth rate is increased relative to the Newtonian case. For all but a narrow range of δ -values (see Fig. 12 (d)), the presence of the yield stress also increases the cutoff wavenumber. Having identified enhanced instability as the salient feature of the non-Newtonian rheology, we turn to *energy-budget analysis* as a means of determining the mechanism for this enhancement.

The linearized dynamical equations associated with the Orr–Sommerfeld equations possess an energy

$$\underbrace{\frac{1}{2} \int_{-\delta}^{0} \mathrm{d}z \int_{0}^{2\pi/\alpha} \mathrm{d}x \, |\boldsymbol{u}_{B}|^{2}}_{=E_{B}} + \underbrace{\frac{1}{2} \int_{0}^{1} \mathrm{d}z \int_{0}^{2\pi/\alpha} \mathrm{d}x \, |\boldsymbol{u}_{T}|^{2}}_{=E_{T}},$$

where $r_B = r$ and $r_T = 1$, which grows or decays in time according to the stability of the base state. By matching the time change in the kinetic energy $\text{KIN}_j = dE_j/dt$ with the inputs of power into the perturbations, an energy budget is obtained [5]:

$$KIN_B + KIN_T = DISS_B + DISS_T + REY_B + REY_T + NOR + TAN,$$

where DISS denotes energy loss through viscous dissipation, REY denotes transfers of energy from the mean flow into the perturbations in the bulk parts of the two phases, while NOR and TAN denote energy delivered by the normal and tangential stresses at the interface. Of particular interest in the present applications are the terms

$$\operatorname{REY}_{T} = \int_{0}^{h} \mathrm{d}z \,\tau_{T,\mathrm{wrs}}\left(z\right) \frac{\mathrm{d}U_{T}}{\mathrm{d}z}, \qquad \tau_{T,\mathrm{wrs}}\left(z\right) = \int_{0}^{2\pi/\alpha} \mathrm{d}x \, uw, \tag{26}$$

and

$$\mathrm{TAN} = \left[\frac{\mathrm{d}U_B}{\mathrm{d}z} \left(m\left(\frac{\mathrm{d}U_B}{\mathrm{d}z}\right)^{n-1} - Bn\left(\frac{\mathrm{d}U_B}{\mathrm{d}z}\right)^{-1} - 1\right)\right]_{z=0} \int_0^{2\pi/\alpha} \mathrm{d}x \,\eta\left(x\right) T_{B,xz}\left(x, z=0\right),$$

where $T_{B,xz}$ is the off-diagonal stress term in the bottom layer. These relationships are derived rigorously in Appendix C. Using this decomposition, we characterize the instability in Figs. 8 and 9, and show the results for various Bingham numbers and flow indices in Tabs. I and II respectively. The character of the instability does not change upon changing the rheological parameters or the aspect ratio δ . In each case considered, the instability is due to the TAN and REY_T terms, the latter being the dominant term. The TAN term is positive when the viscosity in the lower liquid exceeds that of the upper liquid, which gives rise to a positive amount of work done by the tangential stress on the interface. Thus, we can unambiguously say that the TAN term arises from a viscosity mismatch across the interface. On the other hand, there are several mechanisms that work to produce a positive REY_T term. To pinpoint which one is at work, we examine the wave Reynolds stress function defined by Eq. (26), for various Bn- and n-values, shown in Fig. 13. No energy transfer from the critical layer (the zone where $U(z) = c_r$) is evident from the wave Reynolds stress plots: indeed, the wave speed c_r lies inside the liquid in (c) and (d). Thus, the positive contribution to the energy budget from REY_T is related to the viscosity mismatch across the interface. Further vindication of this claim can be found by examination of Fig. 3 in [5].



FIG. 12: Comparison between the maximum growth rate for a non-Newtonian fluid (Bn = 0.3, n = 1), and Newtonian fluid, where $Re_0=1000$. (a) and (b) show the effect of varying the density ratio, at $\delta = 0.2$ and m = 10. Both the maximum growth rate and the cutoff are shifted to higher values for the non-Newtonian fluid. (c) and (d) show the effect of varying the aspect ratio, for r = 2 and m = 10; lastly, (e) and (f) show the effect of varying the viscosity contrast m for r = 2 and $\delta = 0.2$.



FIG. 13: Streamfunction and wave Reynolds stress plots respectively. (a) and (b) show results for n = 1, Bn = 0.25, and $\delta = 0.5$; (c) and (d) show results for n = 0.8, Bn = 0, and $\delta = 0.05$. In the first case, $c_r - U(0) = 1.8$, a small value which gives rise to a critical layer in the linear region of the flow profile $U_T(z)$; in the second case, $c_r - U(0) = -8.76$, which gives rise to a critical layer in the liquid. These results rule out the possibility that the critical layer mechanism plays any role in the formation of the instability.

V. ADDITIONAL MODES OF INSTABILITY

As in the laminar case, a second mode of instability exists for high Reynolds numbers. There, however, ends the similarity with laminar flows. In the situation we consider here, this 'internal' mode is due to stresses in the bottom layer, rather than the top layer. Thus, increasing the Bingham number is stabilizing, since it leads to a higher effective viscosity in the liquid. Similarly, increasing the flow index is stabilizing. Indeed, for shear-thinning modes, two internal modes become positive, although even in this case, the possibility of mode competition is ruled out, since the interfacial mode dominates. In this section we give these ideas some clarity by examining the growth rates, energy budgets, and streamfunctions of this second mode.

Bn	α_{\max}	KIN_T	KIN_B	REY_B	REY_T	DISS_B	DISS_T	NOR	TAN
0	36	0.37	0.63	0.00	1.04	-0.23	-0.28	-0.02	0.49
0.1	34	0.36	0.64	0.00	1.03	-0.20	-0.25	-0.01	0.44
0.3	36	0.37	0.63	0.00	1.03	-0.20	-0.25	-0.01	0.44
Bn	α_{\max}	KIN_T	KIN_B	REY_B	REY_T	$DISS_B$	$DISS_T$	NOR	TAN
0	29	0.34	0.66	-0.01	1.02	-0.17	-0.25	-0.02	0.43
0.1	28	0.35	0.65	-0.01	1.01	-0.14	-0.21	-0.01	0.36
0.3	31	0.50	0.50	0.00	1.04	-0.24	-0.21	-0.01	0.43
Bn	α_{\max}	KIN_T	KIN_B	REY_B	REY_T	DISS_B	DISS_T	NOR	TAN
0	17	0.34	0.66	-0.01	1.00	-0.11	-0.16	-0.02	0.30
0.1	20	0.36	0.64	-0.01	1.01	-0.13	-0.18	-0.01	0.33
0.3	23	0.36	0.64	0.00	1.02	-0.14	-0.20	-0.01	0.33

TABLE I: Energy budget as a function of Bingham number Bn for the most dangerous mode with $\delta = 0.05$, $\delta = 0.2$, and $\delta = 0.5$.

n	α_{\max}	KIN_T	KIN_B	REY_B	REY_T	DISS_B	DISS_T	NOR	TAN	
0.8	38	0.40	0.60	0.00	1.00	-0.12	-0.22	-0.02	0.37	
1.0	36	0.37	0.93	0.00	1.04	-0.23	-0.28	-0.02	0.49	
1.4	29	0.21	0.79	-0.01	1.30	-0.89	-0.93	-0.02	1.55	
n	α_{\max}	KIN_T	KIN_B	REY_B	REY_T	$DISS_B$	$DISS_T$	NOR	TAN	
0.8	31	0.38	0.62	0.00	0.99	-0.10	-0.19	-0.02	0.32	
1.0	29	0.34	0.66	-0.01	1.02	-0.17	-0.25	-0.02	0.43	
1.4	19	0.23	0.77	-0.04	1.10	-0.28	-0.29	-0.01	0.52	
n	$\alpha_{\rm max}$	KIN_T	KIN_B	REY_B	REY_T	DISS_B	DISS_T	NOR	TAN	
0.8	21	0.38	0.62	-0.01	0.99	-0.09	-0.17	-0.02	0.29	
1.0	17	0.34	0.66	-0.01	1.00	-0.11	-0.16	-0.01	0.30	
1.4	12	0.26	0.74	-0.05	1.08	-0.25	-0.19	-0.01	0.41	

TABLE II: Energy budget as a function of flow index n for the most dangerous mode with $\delta = 0.05$, $\delta = 0.2$, and $\delta = 0.5$.

A dispersion curve demonstrating the development of a second unstable mode is shown in Fig. 14. This mode appears at large Reynolds numbers, as evidenced by the consideration of the $Re_0 = 5000$, n = 1, Bn = 0 case. The growth rate of this mode is reduced by the addition of the yield stress, since this addition effectively increases the viscosity in the liquid layer: a similar result holds for the shear-thickening fluid. On the other hand, the growth rate is increased for the shear-thinning fluid. In fact, *two* unstable modes appear in this case (n = 0.8), as evidenced in Fig. 15. The energy source of these modes is threefold, coming from the interface, and the bulk flow in both layers: see Tab. III. Nevertheless, we call these modes "internal", since their critical layer lies deep inside the bottom layer: $c_r < U(0)$ in all cases considered; for further evidence of this fact, see the bottom-layer maxima in the wave

n	Bn	Type	$\alpha_{\rm max}$	KIN_T	KIN_B	REY_B	REY_T	DISS_B	DISS_T	NOR	TAN
1	0	Internal (1)	2.3	0.57	0.43	2.94	0.16	-1.72	- 6.03	0.00	5.65
1.4	0	Internal (1)	0.8	0.00	1.00	0.03	2.79	-0.19	- 1.64	0.00	0.00
0.8	0	Internal (1)	1.7	0.50	0.50	2.03	-0.47	-0.12	-10.60	0.02	11.16
n	Bn	Type	$\alpha_{\rm max}$	KIN_T	KIN_B	REY_B	REY_T	$DISS_B$	$DISS_T$	NOR	TAN
0.8	0	Internal (2)	6.1	0.11	0.89	0.03	0.54	-0.27	-2.63	0.00	3.33

TABLE III: Energy budget for the most dangerous mode, $Re_0 = 5000$, $\delta = 0.2$, m = 10, and r = 2, as a function of flow index n. The parametric dependence on the Bingham number is not considered here because the addition of the yield stress stabilizes this mode.



FIG. 14: The existence of a second mode for $Re_0 = 5000$, $\delta = 0.2$, m = 10, and r = 2, as a function of Bingham number and flow index. Increasing the Bingham number stabilizes the mode, as does an increase in the flow index. Reducing the flow index below unity destabilizes the mode.

Reynolds stress functions in Fig. 16. Note finally that in both Figs. 14 and 15 these internal modes have a growth rate that is smaller than that of the interfacial mode by an order of magnitude. Nevertheless, these modes are observable in principle, since their dimensional growth rates can be large [6, 10].

VI. BARELY-YIELDED FLUIDS

In this section we address the effects of the perturbations on a barely-yielded fluid, for which the derivative of the base-state profile in the non-Newtonian fluid approaches zero at the interface. As in Sec. II C, we make use of the regularized Herschel–Bulkley model for convenience, although this does not enter into the linear-stability calculations, since these are carried out above the yield threshold. Rather, the regularized model is used to provide a threshold such that the instability can preciptate the formation of yielded regions near the interface. Again, for simplicity, we focus on Bingham fluids with index n = 1.

First, for a given parameter set below the yield threshold, we investigate the dispersion



FIG. 15: Two internal modes occur for flow index n = 0.8 and Bn = 0. Here, $Re_0 = 5000$, $\delta = 0.2$, m = 10, and r = 2. The internal modes are an order of magnitude smaller than the interfacial mode: here (a) is an inset of (b).



FIG. 16: Streamfunctions (a) and wave Reynolds stress functions (b) for the internal modes (1) and (2) found in the n = 0.8 case. These flow functions are computed for the most dangerous mode. is The wave Reynolds stress function has large positive contributions in both modes, indicating that the source of the instability is in the bottom, non-Newtonian layer. Here Bn = 0, $Re_0 = 5000$, $\delta = 0.2$, m = 10, and r = 2.

curve as a function of the Bingham number. This parameter set is $Re_0 = 1000$, m = 10r = 2, $\delta = 0.1$, and Γ and G fixed according to the discussion after Eq. (25). This is done in Fig. 17. As the Bingham number is increased gradually, the maximum growth rate also increases, consistent with the findings in Sec. IV. However, as the critical Bingham number $Bn_c = Re_*^2/Re_0^2$ is approached, this trend is reversed and the maximum growth rate along this (interfacial) modal branch decreases (Figs. 17 (a) and (b)). At the same time, a second unstable mode comes into existence, and plays the role of the most dangerous mode (c).



FIG. 17: Dispersion curves near the critical Bingham number. (a) The growth rate as function of wave number, with $Re_0 = 1000$, m = 10, and r = 2; (b) the maximum growth rate as a function of Bingham number along the interfacial branch; (c) The maximum growth rate along both the internal and interfacial branches, near the critical Bingham number. Mode competition occurs and the internal mode becomes the most dangerous one.

That this new mode is internal in nature is confirmed by an energy-budget calculation (not shown), and by the fact the associated wave speed is negative. In Fig. 17 (c), the internal mode fails to be stabilized by surface tension at short wavelengths. This result is surprising, and we have ruled out the possibility that it is a numerical artefact arising from the small values of $(dU_B/dz)_{z=0}$ that appear in the problem: by increasing the surface-tension coefficient Γ , while keeping $(dU_B/dz)_{z=0}$ small, this effect vanishes, suggesting that the it is genuine. Moreover, this result indicates the possibility of ill-posedness in the problem. This possibility is discounted however, by examination of the subsequent phase of the wave evolution, whereby these unstable, near-critical waves precipitate the formation of unyielded regions, which then stabilize these fast-growing waves. It is to this question that we now turn.

Within the framework of the regularized model discussed in Sec. IIC, we focus on one



FIG. 18: Summary of initial conditions used to evolve the linear wave in time, and to precipitate the formation of unyielded regions from a fully-yielded base state.

particular border between well-yielded and quasi-yielded flows, for which the cutoff rate of strain Π_c is small but finite, and for which the difference between Π_c and the base-state rate of strain $(dU_B/dz)_{z=0}$ is $O(\varepsilon)$; this relationship is shown schematically in Fig. 18. The initial rate of strain $\Pi(t=0, x, z=0)$ differs from the base-state rate of strain $(dU_B/dz)_{z=0}$ by an amount that is $O(\varepsilon)$, and is chosen such that

$$\max_{x \in [0, 2\pi/\alpha]} \left| \Pi \left(t = 0, x \right) - \frac{\mathrm{d}U_B}{\mathrm{d}z} \right|_{z=0} < \left| \pi_{\mathrm{c}} - \frac{\mathrm{d}U_B}{\mathrm{d}z} \right|_{z=0}$$

Thus, the initial state is well-yielded. This particular boundary between well- and quasiyielded flows is straightforward to deal with in a linear-stability analysis, since the total rate of strain

$$\Pi = \sqrt{\left(E_{ij}^{(0)} + \delta E_{ij}\right) \left(E^{(0),ij} + \delta E^{ij}\right)},$$
$$= \frac{\mathrm{d}U_B}{\mathrm{d}z} + \delta w_x + \delta u_z$$

can be linearized in the usual manner, without the necessity for a two-parameter expansion in $(dU_B/dz)_{z=0}$ and the wave amplitude ε . With these initial conditions, the location of the yield surface in the base state is at $z = -h_0 = -[m\pi_0 + Bn - (Re_*^2/Re_0^2)] > 0$ (Eq. (16)), that is, no yield surface exists. The perturbation shifts the location of the yield surface from this equilibrium value to a new value

$$h_0 + \delta h_0 = m\pi_0 + Bn - \frac{Re_*^2}{Re_0^2} - \varepsilon \frac{m}{2Re_0} \Re \left[e^{i\alpha x} \left(D^2 + \alpha^2 \right) \phi_B \right]_{z=0},$$
(27)

where ε is the amplitude of the initial interfacial wave. (That the shift should be related to $\Re \left[e^{i\alpha x} \left(D^2 + \alpha^2 \right) \phi_B \right]_{z=0}$ is obvious; the prefactor $m/(2Re_0)$ is computed in Appendix C.)



FIG. 19: Time-evolution of the viscosity field $\mu(x, z, t) / \mu_{\text{cutoff}}$ for (a) t = 0; (b) t = 0.012; (c) t = 0.024; (d) t = 0.036. The interface is given an exaggerated elevation to demonstrate its phase relationship with the viscosity, and the parameter values are $\alpha = 33$, $\delta = 0.1$, $Re_0 = 1000$, n = 1, m = 10, r = 2; the rest of the values are given in Eq. (28).

Such a wave can therefore cause a localized downward shift in the location of the yield surface, if it exists, or, by the same reasoning, the creation of a yield surface in zones for which $h_0 + \delta h_0 < 0$. This is expected at x-values for which $\Re [e^{i\alpha x} (D^2 + \alpha^2) \phi_B]_{z=0} < 0$. We apply some numerical values to the schematically-depicted initial conditions in Fig. 18. Specifically,

$$\alpha = 33,
\pi_{c} = 10^{-4},
Bn = 0.9999 \times 0.48995498,
\varepsilon = 10^{-3} \left| \frac{\pi_{c} - (dU_{B}/dz)_{z=0}}{\max_{x \in [0, 2\pi/\alpha]} \left[\Re \left(e^{i\alpha x} \left(D^{2} + \alpha^{2} \right) \phi_{B} \right)_{z=0} \right]} \right|.$$
(28)

Fig. 19 shows four snapshots in time of the viscosity field in the non-Newtonian layer, with an artificially large interfacial wave shown for illustrative purposes. The viscosity is seen to grow in magnitude in the trough of the wave.

We also plot the phase relationship between the stress function $\tau(x) = (m/Re_0) \Re [e^{i\alpha x} (D^2 + \alpha^2) \phi_B]_{(t=0,z=0)}$ and the interfacial elevation in Fig. 20. The maxima and minima of these functions are almost in phase: peaks and troughs of η and τ almost



FIG. 20: Phase relationship between the interfacial wave and the stress function $\tau(x) = (m/Re_0) \Re \left[e^{i\alpha x} \left(D^2 + \alpha^2 \right) \phi_B \right]_{(t=0,z=0)}$, for $\alpha = 33$. Here $\delta = 0.1$, $Re_0 = 1000$, n = 1, m = 10, r = 2; the other parameter values are given in Eq. (28).

coincide. Thus, as the limiting viscosity is attained (as in Fig. 19), a yield surface must be introduced into the problem, at a location given by Eq. (27). According to Eq. (27) and Fig. 20 the yield surface and the interface will enclose a small unyielded region centred around the trough of the free-surface wave. Now the analysis of the system beyond the time where this yield surface suddenly forms is not possible, and full numerical simulations are required. Nevertheless, our linearized study has produced one quantitative prediction, namely the formation of unyielded regions of fluid in the troughs of the interfacial wave.

An equivalent, and more concrete way of demonstrating this result is to compute the value of the wave amplitude $\varepsilon e^{\lambda_r t} |\cos [\alpha (x - c_r t)]|$ for which the rate of strain attains the cutoff value. This value of the wave amplitude we call A, and is a function of Bingham number. If the A-value can be made to be $O(\varepsilon)$, then the linearized dynamics will precipitate the formation of unyielded regions. A plot of the A-value as a function of Bingham number will thus demonstrate if this is possible. The A-value can be calculated by equating the total rate of strain

$$\Pi = \Pi^{(0)} + \delta w_x + \delta u_z = \Pi^{(0)} + \varepsilon e^{\lambda_r t} \cos\left[\alpha \left(x - c_r t\right) + \theta\right] \left(D^2 + \alpha^2\right) \Psi(z),$$

with the cutoff value:

$$\varepsilon e^{\lambda_{\rm r} t} \cos\left[\alpha \left(x - c_{\rm r} t\right) + \theta\right] = \frac{\pi_{\rm c} - (\mathrm{d} U_B/\mathrm{d} z)_{z=0}}{\left[\left(\mathrm{D}^2 + \alpha^2\right)\Psi\right]_{z=0}} \equiv A.$$

We carry out a numerical computation of A(Bn) for the following parameter set:

$$\pi_{\rm c} = 10^{-4},$$

$$Bn_{\rm c} = 0.48995498, \qquad ({\rm d}U_B/{\rm d}z)_{z=0,Bn=Bn_{\rm c}} = \pi_{\rm c},$$

$$\alpha = 33,$$

The result is shown in Fig. 21. Varying the cutoff rate of strain π_c between $10^{-3}-10^{-5}$ has a negligible effect on this curve, and this parameter change is not displayed. As the critical Bingham number is approached, A tends to zero. Thus, for sufficiently large Bingham numbers, A is $O(\varepsilon)$, the critical rate of strain is attained, and unyielded regions form.



FIG. 21: The critical wave amplitude A for the formation of unyielded regions, as a function of Bingham number, at $\alpha = 33$. As the critical Bingham number is approached, the critical wave amplitude tends to zero, indicating that small-amplitude waves can precipitate the formation of unyielded regions, provided the Bingham number is sufficiently close to criticality.

VII. CONCLUSIONS

We have formulated a linear theory that describes waves that develop at the interface between a bottom layer of non-Newtonian fluid and a top layer that is Newtonian in nature, but is fully developed and turbulent. This model has two components. The first is a base state describing the flat-interface state, which takes account of the non-Newtonian rheology in the bottom layer, and provides the velocity profile and a means of evaluating the interfacial shear stress as a function of the pressure gradient and other fluid parameters. The other component is a linear-stability analysis based on the Orr–Sommerfeld formalism that predicts when the base state is linearly stable. The results of this analysis are summarised in dispersion curves, where the growth rate is given as a function of the disturbance wavenumber. This enables a parametric study, where we found that increasing the Bingham number in the bottom layer is destabilizing, but increasing the flow index is stabilizing if the imposed driving pressure gradient (or else the flow rate) is kept constant. Increasing the Bingham number also brings a broader spectrum of modes into play, since the cutoff wavenumber is shifted to higher values; the dependence of the cutoff on the flow index n is non-monotonic however, with a minimum that strongly depends on the film thickness.

This first parameter study is carried out for moderate values of Bingham number, in the sense that the Bingham number is not sufficiently large to introduce unyielded regions into the base state. In fact, there is a critical Bingham number below which such unyielded regions do not form. Nevertheless, the near-critical region of parameter space is of interest, and thus in Sec. VI we address the notion of *barely yielded* flows, that is, flows for which the base state is unyielded but near criticality. For these flows, we have demonstrated that for Bingham numbers sufficiently close to criticality, linear waves can precipitate the formation of unyielded regions, and that these regions form in the troughs of the waves adjacent to the interface. This implies that current numerical methods for direct numerical simulations must resolve the jump in effective viscosity across the interface in a sharp manner, such

the application of the ghost-fluid methods [30] to two-phase incompressible flow (e.g., the work of Desjardins *et al.* [31]). Otherwise, the shape of high-viscosity regions adjacent to the interface would be much distorted by any smoothing.

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APPENDIX A

In this section, we derive the interfacial conditions involving the perturbed stress tensor

$$\begin{split} \mathsf{T}_{xx,B} &= -p + \frac{2\mu_0}{Re_0} \frac{\partial u}{\partial x}, \qquad \mathsf{T}_{zz,B} = -p + \frac{2\mu_0}{Re_0} \frac{\partial w}{\partial z}, \\ \mathsf{T}_{xz,B} &= \frac{1}{Re_0} \left(\mu_0 + \beta_0 \frac{\mathrm{d}U_B}{\mathrm{d}z} \right) \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \end{split}$$

$$\mathsf{T}_{xx,T} = -p + \frac{2}{Re_0} \frac{\partial u}{\partial x}, \qquad \mathsf{T}_{zz,T} = -p + \frac{2}{Re_0} \frac{\partial w}{\partial z}, \qquad \mathsf{T}_{xz,T} = \frac{1}{Re_0} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right).$$

We shall find the following notation for the viscosity helpful:

$$\mathcal{M}_B = rac{\mu_0}{Re_0}, \qquad \mathcal{M}_T = rac{1}{Re_0}.$$

At the interface $z = \eta(x, t)$, the components of this tensor that make physical sense are the shear and normal stresses, respectively

Shear stress =
$$\hat{s}_{\alpha}\hat{n}_{\beta}\mathsf{T}_{\alpha\beta}$$
, Normal stress = $\hat{n}_{\alpha}\hat{n}_{\beta}\mathsf{T}_{\alpha\beta}$,

where we use the summation convention and sum over repeated indices. The vector \hat{s} is a unit vector parallel to the perturbed interface, while \hat{n} is a unit vector perpendicular to \hat{s} .

The shear-stress condition: The shear stress on either side of the interface is the same, that is,

$$\hat{s}_{\alpha}\hat{n}_{\beta}\left(\mathsf{T}_{\alpha\beta}^{T}-\mathsf{T}_{\alpha\beta}^{B}\right)=0,\qquad\text{sum over }\alpha,\beta.$$

Now to lowest order, we have

$$\hat{\boldsymbol{n}} = (-\eta_x, 1) + O\left(\eta^2\right), \qquad \hat{\boldsymbol{s}} = (1, \eta_x) + O\left(\eta^2\right),$$

with

$$\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{s}}=0$$

Now

$$\begin{aligned} \hat{\boldsymbol{s}}_{\alpha} \hat{\boldsymbol{n}}_{\beta} \mathsf{T}_{\alpha\beta} &= -\eta_{x} \mathsf{T}_{xx} + \eta_{x} \mathsf{T}_{zz} + \mathsf{T}_{xz} + \eta_{x}^{2} \mathsf{T}_{xz}, \\ &\approx \eta_{x} \left(\mathsf{T}_{zz}^{(0)} - \mathsf{T}_{xx}^{(0)} \right) + \mathsf{T}_{xz}^{(0)} + \mathsf{T}_{xz}^{(1)} + \mathrm{H.O.T.}, \end{aligned}$$

Since $\mathsf{T}_{zz}^{(0)} = \mathsf{T}_{xx}^{(0)} = -P_0$, this condition reduces to

$$\left[\left[\mathsf{T}_{xz}^{(0)}(0) + \frac{\mathrm{d}\mathsf{T}_{xz}^{(0)}}{\mathrm{d}z}(0)\,\eta + \mathsf{T}_{xz}^{(1)}(0) + \mathrm{H.O.T.} \right] \right] = 0,$$

where $[[\cdot]]$ is the jump. Since the stresses are matched at zeroth order, it suffices to consider the jump condition

$$\left[\left[\frac{\mathrm{d}\mathsf{T}_{xz}^{(0)}}{\mathrm{d}z} \left(0 \right)\eta + \mathsf{T}_{xz}^{(1)} \left(0 \right) \right] \right] = 0,$$

which reduces to

$$\left(\mathrm{D}^{2}+\alpha^{2}\right)\phi_{T}=\mu_{0}\left(\mathrm{D}^{2}+\alpha^{2}\right)\phi_{B}+\eta\left(\mu_{0}\frac{\mathrm{d}^{2}U_{B}}{\mathrm{d}z^{2}}+\frac{\mathrm{d}\mu_{0}}{\mathrm{d}z}\frac{\mathrm{d}U_{B}}{\mathrm{d}z}-\frac{\mathrm{d}^{2}U_{T}}{\mathrm{d}z^{2}}\right)+\beta_{0}\frac{\mathrm{d}U_{B}}{\mathrm{d}z}\pi_{B}$$

where $\eta = \phi_B(0) / (c - U_B)$ is the interface height,

$$\frac{\mathrm{d}}{\mathrm{d}z} \left(\mu_0 \frac{\mathrm{d}U_B}{\mathrm{d}z} \right) - \frac{\mathrm{d}^2 U_T}{\mathrm{d}z^2} = 0,$$

and

$$\pi_B = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \left(\mathbf{D}^2 + \alpha^2\right)\phi_B.$$

Hence, we obtain the third interfacial condition,

$$\left(\mathbf{D}^{2} + \alpha^{2}\right)\phi_{T} = \left(\mu_{0} + \beta_{0}\frac{\mathrm{d}U_{B}}{\mathrm{d}z}\right)\left(\mathbf{D}^{2} + \alpha^{2}\right)\phi_{B}.$$
(A-1)

The normal-stress condition: The difference between the normal stress across the interface is related to the surface tension through the formula

$$\hat{n}_{\alpha}\hat{n}_{\beta}\left(\mathsf{T}_{\alpha\beta}^{T}-\mathsf{T}_{\alpha\beta}^{B}\right)=-\Gamma_{0}K,\qquad\text{sum over }\alpha\text{ and }\beta.$$

where K is the curvature of the interface, and in the linear approximation is simply

$$K = \eta_{xx}.$$

The constant $\Gamma_0 = \sigma / (\rho_G H U_0^2)$ is the non-dimensional surface tension corresponding to the dimensional surface tension σ Thus, $[[\hat{n}_{\alpha}\hat{n}_{\beta}\mathsf{T}_{\alpha\beta}]] = -\Gamma_0\eta_{xx}$ on $z = \eta$. Hence,

$$\left[\left[2\hat{n}_x\hat{n}_z\mathsf{T}_{xz}+\hat{n}_z\hat{n}_z\mathsf{T}_{zz}+\hat{n}_x\hat{n}_x\mathsf{T}_{xx}\right]\right]=-\Gamma_0\eta_{xx}$$

But $\hat{n}_x \hat{n}_x = \eta_x^2$, while $\hat{n}_z \hat{n}_z = 1$ and thus, to lowest order in η and its derivatives, the normal stress is $\mathsf{T}_{zz}^{(0)} + \mathsf{T}_{zz}^{(1)} + 2\eta_x \mu \, (\mathrm{d}U/\mathrm{d}z)$, and the jump condition is

$$\left[\left[-2\eta_x \mathcal{M} \frac{\mathrm{d}U}{\mathrm{d}z} + \mathsf{T}_{zz}^{(0)} + \mathsf{T}_{zz}^{(1)} \right] \right]_B^T = -\Gamma_0 \eta_{xx},$$

where $\Gamma_0 = \sigma / (\rho_T H U_0^2)$ is the non-dimensional surface tension corresponding to the dimensional surface tension σ , and η_{xx} is the linearized curvature. Since the stresses $\mathcal{M}(dU/dz)$ and $\mathsf{T}_{zz}^{(0)}$ are matched at z = 0, it suffices to consider the condition

$$\left[\left[\eta \frac{\mathrm{d}\mathsf{T}_{zz}^{(0)}}{\mathrm{d}z} + \mathsf{T}_{zz}^{(1)} \right] \right]_B^T = -\Gamma_0 \eta_{xx}$$

Now $T_{zz}^{(0)} = -P$, and $[[r]]_B^T = (r_T - r_B)$. Hence, by hydrostatic balance, the normal-stress condition is reduced to

$$\begin{bmatrix} \begin{bmatrix} \mathsf{T}_{zz}^{(1)} \end{bmatrix} \end{bmatrix}_{B}^{T} = -\Gamma_{0}\eta_{xx} - G_{0}(r_{T} - r_{B})\eta, = \begin{bmatrix} \Gamma_{0}\alpha^{2} + (r_{B} - r_{T})G_{0} \end{bmatrix}\eta,$$

where $G_0 = gH/U_0^2$. Thus,

$$-(p_T - p_B) - \frac{2i\alpha}{Re_0} \left(\mathbf{D}\phi_T - \mu_0 \mathbf{D}\phi_B \right) = \left[\Gamma_0 \alpha^2 + (r_B - r_T) G_0 \right] \eta.$$

In the upper layer,

$$p_T = \frac{1}{i\alpha Re_0} D^3 \phi_T + \left[\left(c - U_T \left(z \right) \right) + \frac{i\alpha}{Re_0} \right] D\phi_T + \phi_T \frac{\mathrm{d}U_T}{\mathrm{d}z},$$

while in the bottom layer,

$$p_{B} = \frac{\mu_{0}}{i\alpha Re_{0}} D^{3} \phi_{B} + \left[r \left(c - U_{B} \left(z \right) \right) + \frac{i\alpha\mu_{0}}{Re_{0}} \right] D\phi_{B} + r\phi \frac{\mathrm{d}U_{B}}{\mathrm{d}z} + \frac{1}{i\alpha Re_{0}} \frac{\mathrm{d}\mu_{0}}{\mathrm{d}z} \left(D^{2} + \alpha^{2} \right) \phi_{B} + \frac{\beta_{0}}{i\alpha Re_{0}} \frac{\mathrm{d}U_{B}}{\mathrm{d}z} D\pi_{B} + \frac{1}{i\alpha Re_{0}} \left(\beta_{0} \frac{\mathrm{d}^{2}U_{B}}{\mathrm{d}z^{2}} + \frac{\mathrm{d}\beta_{0}}{\mathrm{d}z} \frac{\mathrm{d}U_{B}}{\mathrm{d}z} \right) \pi_{B}.$$

Thus, the normal-stress condition is

$$r\left[\left(c-U_{B}\right)\mathrm{D}\phi_{B}+\frac{\mathrm{d}U_{B}}{\mathrm{d}z}\phi_{B}\right]-\left[\left(c-U_{T}\right)\mathrm{D}\phi_{T}+\frac{\mathrm{d}U_{T}}{\mathrm{d}z}\phi_{T}\right]$$
$$+\frac{\mu_{0}}{i\alpha Re_{0}}\left(\mathrm{D}^{3}\phi_{B}-3\alpha^{2}\mathrm{D}\phi_{B}\right)-\frac{1}{i\alpha Re_{0}}\left(\mathrm{D}^{3}\phi_{T}-3\alpha^{2}\mathrm{D}\phi_{T}\right)$$
$$+\frac{1}{i\alpha Re_{0}}\frac{\mathrm{d}\mu_{0}}{\mathrm{d}z}\left(\mathrm{D}^{2}+\alpha^{2}\right)\phi_{B}+\frac{\beta_{0}}{i\alpha Re_{0}}\frac{\mathrm{d}U_{B}}{\mathrm{d}z}\mathrm{D}\pi_{B}+\frac{1}{i\alpha Re_{0}}\left(\beta_{0}\frac{\mathrm{d}^{2}U_{B}}{\mathrm{d}z^{2}}+\frac{\mathrm{d}\beta_{0}}{\mathrm{d}z}\frac{\mathrm{d}U_{B}}{\mathrm{d}z}\right)\pi_{B}$$
$$=\left[\Gamma_{0}\alpha^{2}+\left(r_{B}-r_{T}\right)G_{0}\right]\eta.$$

This condition is tidied up by taking $\Gamma = \Gamma_0 Re_0$, $G = (r_B - r_T) G_0 Re_0$. Additionally, we use the free-surface condition $\eta = \phi_B(0) / (c - U_B)$. Hence,

$$i\alpha Re_0 r \left[(c - U_B) D\phi_B + \frac{\mathrm{d}U_B}{\mathrm{d}z} \phi_B \right] + \mu_0 \left(D^3 \phi_B - 3\alpha^2 D\phi_B \right) + \frac{\mathrm{d}\mu_0}{\mathrm{d}z} \left(D^2 + \alpha^2 \right) \phi_B + \beta_0 \frac{\mathrm{d}U_B}{\mathrm{d}z} D\pi_B + \left(\beta_0 \frac{\mathrm{d}^2 U_B}{\mathrm{d}z^2} + \frac{\mathrm{d}\beta_0}{\mathrm{d}z} \frac{\mathrm{d}U_B}{\mathrm{d}z} \right) \pi_B = i\alpha Re_0 \left[(c - U_T) D\phi_T + \frac{\mathrm{d}U_T}{\mathrm{d}z} \phi_T \right] + \left(D^3 \phi_T - 3\alpha^2 D\phi_T \right) + i\alpha \left(\Gamma \alpha^2 + G \right) \frac{\phi_B}{c - U_B}.$$

Lastly, identifying $\pi_B = (D^2 + \alpha^2) \phi_B$, we have the normal-stress condition

$$i\alpha Re_0 r \left[(c - U_B) D\phi_B + \frac{\mathrm{d}U_B}{\mathrm{d}z} \phi_B \right] + \mu_0 \left(D^3 \phi_B - 3\alpha^2 D\phi_B \right) + b_0 (z) \phi_B + b_1 (z) D\phi_B + b_2 (z) D^2 \phi_B + b_3 (z) D^3 \phi_B = i\alpha Re_0 \left[(c - U_T) D\phi_T + \frac{\mathrm{d}U_T}{\mathrm{d}z} \phi_T \right] + \left(D^3 \phi_T - 3\alpha^2 D\phi_T \right) + i\alpha \left(\Gamma \alpha^2 + G \right) \frac{\phi_B}{c - U_B},$$

where

$$b_0(z) = \alpha^2 \left[\frac{\mathrm{d}\mu_0}{\mathrm{d}z} + \beta_0 \frac{\mathrm{d}^2 U_B}{\mathrm{d}z^2} + \frac{\mathrm{d}\beta_0}{\mathrm{d}z} \frac{\mathrm{d}U_B}{\mathrm{d}z} \right],$$

$$b_1(z) = \alpha^2 \beta_0 \frac{\mathrm{d}U_B}{\mathrm{d}z},$$

$$b_2(z) = \frac{\mathrm{d}\mu_0}{\mathrm{d}z} + \beta_0 \frac{\mathrm{d}^2 U_B}{\mathrm{d}z^2} + \frac{\mathrm{d}\beta_0}{\mathrm{d}z} \frac{\mathrm{d}U_B}{\mathrm{d}z},$$

$$b_3(z) = \beta_0 \frac{\mathrm{d}U_B}{\mathrm{d}z}.$$

APPENDIX B

To understand the mechanism by which the parametrized stress enhances the growth rate of the wave, we perform an energy-budget analysis. To find the energy budget of the system, we multiply the Reynolds-averaged Navier–Stokes (RANS) equation of the linear problem by \boldsymbol{u}_j , where j = B, T, and integrate over a single wavelength (periodic cell) in the x-direction, and over the entire z-domain.

The linearized RANS equations have the following form:

$$r_j \left(\frac{\partial \boldsymbol{u}_j}{\partial t} + \boldsymbol{U}_j \cdot \nabla \boldsymbol{u}_j + \boldsymbol{u}_j \cdot \nabla \boldsymbol{U}_j \right) = \nabla \cdot \mathsf{T}_j, \qquad (B-1a)$$

$$\nabla \cdot \boldsymbol{u}_j = 0, \tag{B-1b}$$

where $(r_B, r_T) = (r = \rho_B / \rho_T, 1)$. The stress terms are given by the formulas

$$\begin{split} \mathsf{T}_{xx,B} &= -p + \frac{2\mu_0}{Re_0} \frac{\partial u}{\partial x}, \qquad \mathsf{T}_{zz,B} = -p + \frac{2\mu_0}{Re_0} \frac{\partial w}{\partial z}, \\ \mathsf{T}_{xz,B} &= \frac{1}{Re_0} \left(\mu_0 + \beta_0 \frac{\mathrm{d}U_B}{\mathrm{d}z} \right) \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \end{split}$$

$$\mathsf{T}_{xx,T} = -p + \frac{2}{Re_0} \frac{\partial u}{\partial x}, \qquad \mathsf{T}_{zz,T} = -p + \frac{2}{Re_0} \frac{\partial w}{\partial z}, \qquad \mathsf{T}_{xz,T} = \frac{1}{Re_0} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right).$$

In deriving the Orr–Sommerfeld equation, we find that the pressure terms have a representation in terms of the stream function $\phi(z)$,

$$p_{B} = \frac{\mu_{0}}{i\alpha Re_{0}} D^{3} \phi_{B} + \left[r \left(c - U_{B} \left(z \right) \right) + \frac{i\alpha\mu_{0}}{Re_{0}} \right] D\phi_{B} + r\phi_{B} \frac{\mathrm{d}U_{B}}{\mathrm{d}z} + \frac{1}{i\alpha Re_{0}} \frac{\mathrm{d}\mu_{0}}{\mathrm{d}z} \left(D^{2} + \alpha^{2} \right) \phi_{B} + \frac{\beta_{0}}{i\alpha Re_{0}} \frac{\mathrm{d}U_{B}}{\mathrm{d}z} D\pi_{B} + \frac{1}{i\alpha Re_{0}} \left(\beta_{0} \frac{\mathrm{d}^{2}U_{B}}{\mathrm{d}z^{2}} + \frac{\mathrm{d}\beta_{0}}{\mathrm{d}z} \frac{\mathrm{d}U_{B}}{\mathrm{d}z} \right) \pi_{B},$$

$$\pi_B = \frac{\partial u_B}{\partial z} + \frac{\partial w_B}{\partial x},\tag{B-2a}$$

$$p_T = \frac{1}{i\alpha Re_0} D^3 \phi_T + \left[(c - U_T(z)) + \frac{i\alpha}{Re_0} \right] D\phi_T + \phi_T \frac{\mathrm{d}U_T}{\mathrm{d}z}, \qquad (B-2b)$$

We multiply (B-1a) by \boldsymbol{u}_j and integrate over the unit cell, $[0, \ell] \times [-d_B, 0]$ for the liquid, and $[0, \ell] \times [0, h]$ for the upper layer. Here $\ell = 2\pi/\alpha$ is the wavelength of the disturbance (working with a sinusoidal disturbance gives periodic boundary conditions in the lateral direction). Using Gauss' theorem on the stress term, we obtain the energy relation for the liquid:

$$\begin{aligned} r_B \int \mathrm{d}x \int \mathrm{d}z \, \left[\frac{1}{2} \frac{\partial \boldsymbol{u}_B^2}{\partial t} + u_B w_B \frac{\mathrm{d}dU_B}{\mathrm{d}dz} \right] = \\ &- \frac{1}{Re_B} \int \mathrm{d}x \int \mathrm{d}z \, \left[2\mu_0 \left(\frac{\partial u_B}{\partial x} \right)^2 + 2\mu_0 \left(\frac{\partial w_B}{\partial z} \right)^2 + \left(\mu_0 + \beta_0 \frac{\mathrm{d}U_B}{\mathrm{d}z} \right) \left(\frac{\partial u_B}{\partial z} + \frac{\partial w_B}{\partial x} \right)^2 \right] \\ &+ \int \mathrm{d}x \, \left[u_B \mathsf{T}_{B,zx} + w_B \mathsf{T}_{B,zz} \right]_{z=0}, \end{aligned}$$

Note that $\mu_0 + \beta_0 (dU_B/dz) = mn (dU_B/dz)^{n-1}$, so the dissipation term is still sign-definite. Similarly, we have the energy relation for the upper layer:

$$r_T \int dx \int dz \left[\frac{1}{2} \frac{\partial \boldsymbol{u}_T^2}{\partial t} + u_T w_T \frac{dU_T}{dz} \right] = -\frac{1}{Re_B} \int dx \int dz \left[2 \left(\frac{\partial u_T}{\partial x} \right)^2 + \left(\frac{\partial u_T}{\partial z} + \frac{\partial w_T}{\partial x} \right)^2 + 2 \left(\frac{\partial w_T}{\partial z} \right)^2 \right] - \int dx \left[u_T \mathsf{T}_{T,zx} + w_T \mathsf{T}_{T,zz} \right]_{z=0}$$

Adding the two equations together we obtain, in a standard fashion, the energy-budget relation

$$\sum_{i=B,T} \text{KIN}_i = \sum_{i=B,T} \text{REY}_i + \sum_{i=B,T} \text{DISS}_i + \text{INT},$$

where

$$\begin{aligned} \text{KIN}_{i} &= \frac{1}{2} \frac{d}{dt} \int \mathrm{d}x \int \mathrm{d}z \, r_{i} \boldsymbol{u}_{i}^{2}, \\ \text{REY}_{i} &= -r_{i} \int \mathrm{d}x \int \mathrm{d}z \, u_{i} w_{i} \frac{\mathrm{d}U_{i}}{\mathrm{d}z}, \\ \text{DISS}_{T} &= -\frac{1}{Re_{0}} \int \mathrm{d}x \int \mathrm{d}z \, \left[2 \left(\frac{\partial u_{T}}{\partial x} \right)^{2} + \left(\frac{\partial u_{T}}{\partial z} + \frac{\partial w_{T}}{\partial x} \right)^{2} + 2 \left(\frac{\partial w_{T}}{\partial z} \right)^{2} \right], \end{aligned}$$

 $DISS_B =$

$$-\frac{1}{Re_0}\int \mathrm{d}x\int \mathrm{d}z \left[2\mu_0 \left(\frac{\partial u_B}{\partial x}\right)^2 + 2\mu_0 \left(\frac{\partial w_B}{\partial z}\right)^2 + mn\left(\frac{\mathrm{d}U_B}{\mathrm{d}z}\right)^{n-1} \left(\frac{\partial u_B}{\partial z} + \frac{\partial w_B}{\partial x}\right)^2\right].$$

Lastly, the term 'INT' is related to interfacial conditions:

INT =
$$\int dx \left[u_B \mathsf{T}_{B,zx} + w \mathsf{T}_{B,zz} \right]_{z=0} - \int dx \left[u_T \mathsf{T}_{T,zx} + w \mathsf{T}_{T,zz} \right]_{z=0},$$

which is decomposed into normal and tangential contributions,

$$INT = NOR + TAN,$$

where

$$NOR = \int dx \left[w_B \mathsf{T}_{B,zz} - w_T \mathsf{T}_{T,zz} \right]_{z=0},$$

and

$$\mathrm{TAN} = \int \mathrm{d}x \, \left[u_B \mathsf{T}_{B,zx} - u_T \mathsf{T}_{T,zx} \right]_{z=0}.$$

Note that the normal contribution can be further decomposed to highlight the effects of gravity and surface tension,

$$NOR = TEN + HYD = \frac{\Gamma}{Re_0} \int_0^\ell dx \,\eta_{xx} w \,(x, z = 0) + \frac{G}{Re_0} \int_0^\ell dx \,\eta w \,(x, z = 0) \,.$$

APPENDIX C

In this section we study the location of the yield surface, and demonstrate that in the presence of a small-amplitude disturbance, the location of the yield surface in the bottom layer shifts from $z = -h_0$, to a value

$$z = -h_0 - \delta h_0, \qquad h_0 + \delta h_0 = m\pi_0 + Bn - \frac{Re_*^2}{Re_0^2} - \varepsilon \frac{m}{2Re_0} \left[e^{i\alpha x} \left(D^2 + \alpha^2 \right) \phi_B \right]_{z=0}, \quad (C-1)$$

To verify Eq. (C-1), we derive the equation for the isosurface of viscosity when a small sinusoidal wave is introduced into the problem. In the base state, the curve $z = -z_0$ is an isosurface of viscosity, corresponding to the viscosity value $\mu_0 = \mu (-z_0)$. When a disturbance is introduced, the isosurface is shifted to $z = -z_0 + \varepsilon f(x)$, where ε is the magnitude of the disturbance. The isosurface is now determined by the relation

$$\mu^{(0)} \left(-z_0 + \varepsilon f(x) \right) + \mu^{(1)} \left(x, -z_0 + \varepsilon f(x) \right) = \mu_0 = \text{Const.},$$

where the superscripts denote the order of successive terms in the small- ε expansion. By expanding this expression further, we obtain the condition

$$\mu^{(0)}(-z_0) + \varepsilon \frac{\mathrm{d}\mu^{(0)}}{\mathrm{d}z} \bigg|_{z=-z_0} f(x) + \mu^{(1)}(x, -z_0) = \mu_0.$$

Since $\mu^{(0)}(-z_0) = \mu_0$, it follows that

$$f(x) = -\frac{\mu^{(1)}(x, -z_0)}{\frac{d\mu^{(0)}}{dz}}\Big|_{z=-z_0}.$$
 (C-2)

The coefficients in Eq. (C-2) can be computed readily:

$$\mu^{(0)} = m + \frac{Bn}{dU_B/dz},$$

$$\frac{d\mu^{(0)}}{dz} = -\frac{Bn}{(dU_B/dz)^2} \frac{d^2 U_B}{dz^2},$$

$$\frac{d^2 U_B}{dz^2} = -\frac{m}{2Re_0},$$

$$\mu^{(1)} = -\frac{Bn}{(dU_B/dz)^2} \Pi^{(1)},$$

hence

$$f\left(x\right) = \frac{m}{2Re_0}\Pi^{(1)},$$

Thus, the isosurface is now located at

$$z = -z_0 + \varepsilon f(x) = -z_0 + \varepsilon \frac{m}{2Re_0} \Pi^{(1)}.$$
 (C-3)

If the value of the viscosity on the isosurface is in fact the limiting viscosity, then Eq. (C-3) gives the location of the yield surface. Using $\Pi^{(1)} = \Re \left[e^{i\alpha x} \left(\mathbf{D}^2 + \alpha^2 \right) \phi_B \right]$ and $z_0 = h_0$, the shifted location of the yield surface is

$$z = -h_0 + \varepsilon \frac{m}{2Re_0} \Re \left[e^{i\alpha x} \left(\mathbf{D}^2 + \alpha^2 \right) \phi_B \right]_{z=0},$$

where we have projected the final term on to the plane z = 0 since $h_0 = O(\varepsilon)$. Thus, we recover Eq. (C-1).

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